Lurupa

Rigorous Error Bounds in Linear Programming

Christian Keil

Computer Science 3
Technical University Hamburg–Harburg

Algebraic and Numerical Algorithms
and Computer-assisted Proofs, 2005

Outline

Outline

A Representation of a Linear Program

Definition (Linear Program)

Find the optimal value f^* of a linear objective function $c^T x$ subject to

- linear constraints $Ax \leq a,\ Bx = b$ and
- simple bounds $\underline{x} \leq x \leq \overline{x}$.

- Set of feasible points F satisfying constraints and simple bounds
- Can be represented by the tuple $P := (c, A, a, B, b)$ and $\underline{x}, \overline{x}$
- Simple bounds may be infinite

A Lower Bound for the Optimal Value (1)

Theorem (Jansson, 2004)

Given a linear program P and $\underline{x}, \overline{x}$*. If interval vectors* $\mathbf{y} \leq 0, \mathbf{z}$ *satisfy*

- $\exists\, y \in \mathbf{y}, z \in \mathbf{z} : c_j - (A_{:j})^T y - (B_{:j})^T z = 0$ *for free* x_j*, and*
- *the defects*

$$\mathbf{d}_j := c_j - (A_{:j})^T \mathbf{y} - (B_{:j})^T \mathbf{z} \begin{cases} \leq 0 & \text{for} \quad -\infty < x_j \leq \overline{x}_j \\ \geq 0 & \text{for} \quad \underline{x}_j \leq x_j < \infty \end{cases}$$

then

$$\underline{f}^* := \inf\{a^T \mathbf{y} + b^T \mathbf{z} + \sum_{\underline{x}_j \neq -\infty} \underline{x}_j \mathbf{d}_j^+ + \sum_{\overline{x}_j \neq \infty} \overline{x}_j \mathbf{d}_j^- \}$$

is a lower bound for the optimal value.

A Lower Bound for the Optimal Value (2)

- $O(n^2)$ operations for finite simple bounds, coincides with [Neumaier and Shcherbina, 2004]
- Conditions not met (infinite simple bounds) $\Rightarrow$ iterate with perturbed dual constraints
- Works for interval problems **P**

TUHH
Technische Universität Hamburg-Harburg

An Upper Bound for the Optimal Value

- Based on idea from [Krawczyk, 1975] later used and modified [Jansson, 1988], [Hansen and Walster, 1991], [Kearfott, 1994]
- Enclose primal interior point
- Verifies existence of primal feasible solutions in contrast to lower bound

TUHH
Technische Universität Hamburg-Harburg

Outline

A Real World Application Test Set

Netlib comprises $\sim$ 100 problems

- First added in 1988, latest in 1996
- From e.g., stochastic forestry problems, oil refinery problems, flap settings of aircraft, pilot models, scheduling, truss structure, image restoration, industrial production and allocation models, multisector economic planning problems
- 32 to 15695 variables, 27 to 16675 constraints (medium size)
- Established test set for lp algorithms
- [Ordóñez and Freund, 2003]: 71% ill-conditioned

Condition and The Distance to Infeasibility

Definition (Distance to Infeasibility ρ)

Norm of the smallest perturbation of P
that results in an empty feasible set.

ρ_P distance to primal infeasibility

ρ_D distance to dual infeasibility

Definition (Condition)

Scale invariant reciprocal of the minimal distance to infeasibility.

$$C(d) := \frac{\|d\|}{\min\{\rho_P, \rho_D\}}$$

Some Examples of What Can Happen

	ρ_D	ρ_P	$\underline{f}^*$	$\overline{f}^*$	μ
scagr25	0.034646	0.021077	$\surd$	$\surd$	$3.7821e-8$
bnl1	0.106400	0	$\surd$		$7.2244e-8$
stair	0	0.000580		$\surd$	$5.4796e-9$
80bau3b	0	0	$\surd$	$\surd$	$5.6653e-8$
scsd8	1.000000	0.268363	$\surd$		$6.3831e-8$

Some Examples of What Can Happen

	ρ_D	ρ_P	$\underline{f}^*$	$\overline{f}^*$	μ
scagr25	0.034646	0.021077	$\surd$	$\surd$	$3.7821e-8$
bnl1	0.106400	0	$\surd$		$7.2244e-8$
stair	0	0.000580		$\surd$	$5.4796e-9$
80bau3b	0	0	$\surd$	$\surd$	$5.6653e-8$
scsd8	1.000000	0.268363	$\surd$		$6.3831e-8$

Dimensions of Example Problems

	m	p	n	nz
scagr25	171	300	500	1554
bnl1	411	232	1175	5121
stair	147	209	467	3856
80bau3b	2262	0	9799	21002
scsd8	0	397	2750	8584

Overview of the Results

In total 89 problems, 86 with infinite simple bounds

- 35 finite upper bounds for the optimal value

$$\text{med}(\mu(\overline{f}^*, f^*)) = 8.0e-9 \qquad \text{med}(t_{\overline{f}^*}/t_{f^*}) = 5.3$$

- 76 finite lower bounds for the optimal value

$$\text{med}(\mu(\underline{f}^*, f^*)) = 2.2e-8 \qquad \text{med}(t_{\underline{f}^*}/t_{f^*}) = 0.5$$

TUHH
Technische Universität Hamburg-Harburg

Outline

Lurupa's Design Goals

- Rigorous bounds for the optimal value
- Use unmodified solvers plus postprocessing
- Easy to use
- Easily adoptable to different solvers
- Standalone and library version

Lurupa's Internals

- Implemented in ANSI C++
- Builds on interval library PROFIL/BIAS [Knüppel, 1994]
- Computational core
  uses solver modules to interface solvers

Usage of Lurupa

Example (command line)

```
>lurupa -sm /path/to/module \
        -lp /path/to/model -lb -ub
```

Meaning of parameters

- **-sm** specify solver module
- **-lp** specify lp model file
- **-lb** compute lower bound
- **-ub** compute upper bound

Usage of Lurupa

Example (API)

```
Lurupa l;
l.set_solver_module("path/to/module");

FILE *in = fopen("path/to/model", "r");
Lp lp = l.read_lp(in, 0);

double optimal, bound, iterations;
l.solve_lp(lp, optimal);
l.lower_bound(lp, bound, iterations);
l.upper_bound(lp, bound, iterations);
```

Future Work

- Verified condition measures
- Sparse structures (in PROFIL)
- New solver modules
- Certificates of infeasibility, unboundedness
- Verified preprocessing [Fourer and Gay, 1993]

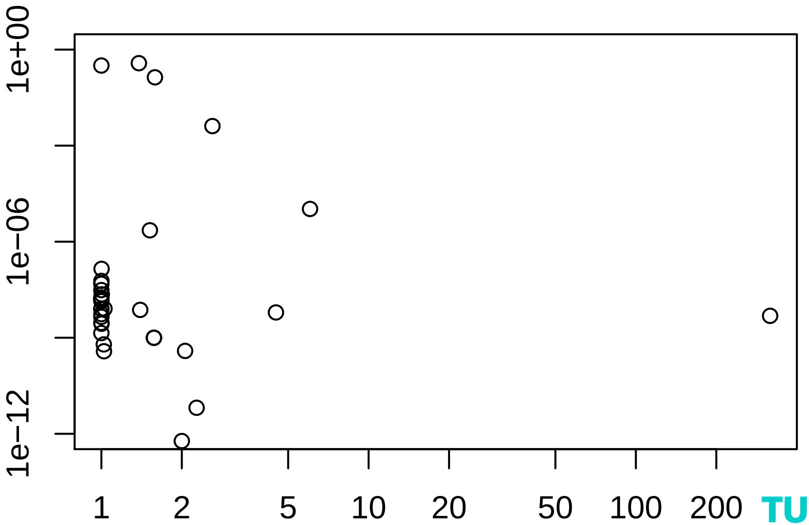
Outline

Bounds and Distances to Infeasibility

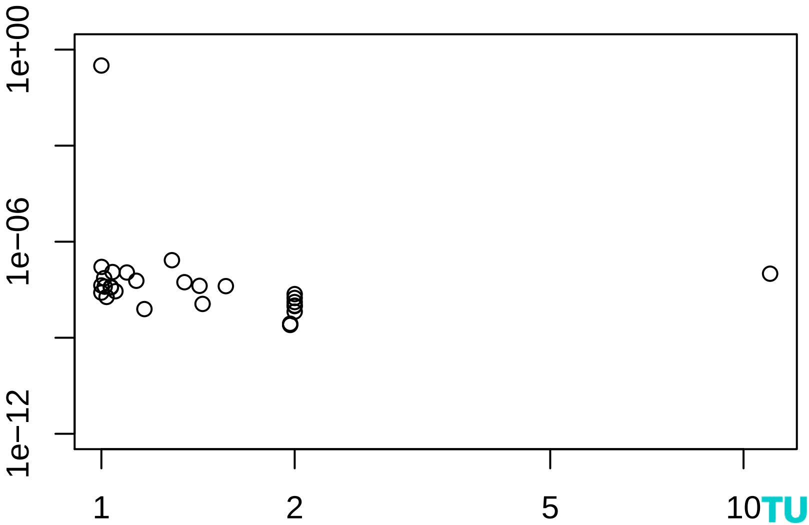
		total	$\rho_P \geq 0$	$\rho_P = 0$	
upper bound	finite	35	26	9	
	infinite	54	2	52	

		total	$\rho_D \geq 0$	$\rho_D = 0$	
lower bound	finite	76	65	10	1 distance NA
	infinite	13	3	10	

$\mu(\overline{f}^*, f^*)$ over ρ_P



$\mu(\underline{f}^*, f^*)$ over ρ_D



Overview of the Results

In total 89 problems, 86 with infinite simple bounds

35 finite upper bounds		$\mu(\overline{f}^*, f^*)$	$t_{\overline{f}^*}/t_{f^*}$
	median	$8.034e-9$	5.250
	mean	$1.481e-2$	38.776

76 finite lower bounds		$\mu(\underline{f}^*, f^*)$	$t_{\underline{f}^*}/t_{f^*}$
	median	$2.183e-8$	0.500
	mean	$7.279e-5$	1.081

32 finite pairs		$\mu(\overline{f}^*, \underline{f}^*)$	$t_{\overline{f}^*}/t_{f^*}$	$t_{\underline{f}^*}/t_{f^*}$
	median	$5.620e-8$	5.000	0.236
	mean	$1.619e-2$	42.317	0.422