Lower bounds for the smallest singular value of certain Toeplitz-like triangular matrices with linearly increasing diagonal entries

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Abstract. Let L be a lower triangular $n \times n$-Toeplitz matrix with first column $(\mu, \alpha, \beta, \alpha, \beta, \dots)^T$, where $\mu, \alpha, \beta \geq 0$ fulfill $\alpha - \beta \in [0, 1)$ and $\alpha \in [1, \mu+3]$. Furthermore let D be the diagonal matrix with diagonal entries $1, 2, \dots, n$. We prove that the smallest singular value of the matrix $A := L + D$ is bounded from below by a constant $\omega = \omega(\mu, \alpha, \beta) > 0$ which is independent of the dimension n.

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In the "Open problems" session of the workshop "Numerical Verification (NIVEA) 2019" in Hokkaido, Yoshitaka Watanabe from Kyushu University posed the problem to bound the smallest singular value of a specific infinite dimensional triangular matrix. The problem arises in the rigorous estimation of an error constant in the Fourier expansion related to numerical verification of the solution of some ordinary differential equation.

In the following we prove such a bound in a more general setting. The original problem is stated as Example 2 at the end of this note.

Let $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, $\mu, \alpha, \beta \in \mathbb{R}$, and let A be the lower triangular matrix with

$$A := \begin{pmatrix} \mu+1 & & & & & & \\ \alpha & \mu+2 & & & & & \\ \beta & \alpha & \mu+3 & & & & \\ \alpha & \beta & \alpha & \mu+4 & & & \\ \beta & \alpha & \beta & \alpha & \mu+5 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ \cdots & \cdots & \beta & \alpha & \beta & \alpha & \mu+n \end{pmatrix}. \qquad (1)$$

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In the mentioned application the question arose under which conditions the smallest singular value σ_n of such Toeplitz-like matrices with linearly increasing diagonal entries is uniformly bounded from below by a positive constant independent of n. We will prove the following theorem.

Theorem 1. *Let $\mu, \alpha, \beta \in \mathbb{R}$ fulfill*

$$0 \le \beta \le \alpha < \beta + 1 \quad \text{and} \quad 1 \le \alpha \le \mu + 3 \quad \text{and} \quad \mu > 0. \tag{2}$$

Then, the matrix A defined in (1) satisfies

$$\|A^{-1}\|_F^{-1} \ \ge \ \sqrt{\frac{\mu+1}{1+\theta(\mu)}} =: \omega \quad \text{where } \theta(\mu) := \frac{\alpha^2\mu(1+4/\mu)^{2-\alpha+\beta}}{(1-\alpha+\beta)(\mu+2)^2} \tag{3}$$

and $\|\cdot\|_F$ denotes the Frobenius norm. Thus, ω is a uniform lower bound for the smallest singular value σ_n of A independent of the dimension n.

Replacing the i-th row of A by the i-th row minus the $(i-2)$-nd row for $i = n, n-1, \ldots, 4$ is performed by multiplying A from the left by the matrix

$$R \ := \left(\begin{array}{c|cccccc} 1 & & & & & & \\ \hline & 1 & & & & & \\ & 0 & 1 & & & & \\ & -1 & 0 & 1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & -1 & 0 & 1 & \end{array}\right). \tag{4}$$

Note that R is the identity matrix if $n \in \{1, 2, 3\}$. For example, for $n = 7$ the row-transformed matrix $\tilde{A} := RA$ reads

$$\tilde{A} \ := \begin{pmatrix} \mu+1 & & & & & & \\ \alpha & \mu+2 & & & & & \\ \beta & \alpha & \mu+3 & & & & \\ & \beta-\mu-2 & \alpha & \mu+4 & & & \\ & & \beta-\mu-3 & \alpha & \mu+5 & & \\ & & & \beta-\mu-4 & \alpha & \mu+6 & \\ & & & & \beta-\mu-5 & \alpha & \mu+7 \end{pmatrix}.$$

The first column c of A^{-1} is the solution of the linear system $Ac = e_1$, where $e_1 = (1, 0, \ldots, 0)^T$ is the first standard basis vector. Since $Re_1 = e_1$, we have $\tilde{A}c = e_1$. i.e., c is also the first column of $\tilde{A}^{-1}$ which computes recursively to

$$c_1 \ = \ \frac{1}{\mu+1} \tag{5}$$

$$c_2 \ = \ \frac{-\alpha c_1}{\mu+2} \tag{6}$$

$$c_3 \ = \ \frac{-\alpha c_2 - \beta c_1}{\mu+3} \tag{7}$$

$$c_i \ = \ \frac{-\alpha c_{i-1} + (\mu+i-2-\beta)c_{i-2}}{\mu+i} \qquad \text{for } i \ge 4. \tag{8}$$

Until further notice we will only use

$$0 \le \beta \le \alpha \le \beta + 2 \qquad \text{and} \qquad 1 \le \alpha \le \mu + 3 \qquad \text{and} \qquad \mu \ge 0 \tag{9}$$

which is weaker than (2). Define

$$\psi_k := \frac{\mu + 2k - 2 + \alpha - \beta}{\mu + 2k} \qquad \text{for } k \in \mathbb{N}_{\ge 2} \tag{10}$$

$$\varphi_k := \prod_{j=2}^{k} \psi_j \qquad \text{for } k \in \mathbb{N}. \tag{11}$$

Note that $\varphi_1 := 1$, by definition of an empty product, and that $\psi_k, \varphi_k \le 1$ by (9). We will now prove by induction that

$$|c_{2k}| \le \varphi_k |c_2| \qquad \text{and} \qquad |c_{2k+1}| \le \varphi_k |c_2| \qquad \text{for all } k \in \mathbb{N}. \tag{12}$$

Since $\varphi_1 = 1$, the left inequality of (12) is trivial for $k = 1$ and the right inequality follows from (6), (7), and (9):

$$\begin{aligned} |c_3| &= \frac{\big|\beta c_1 - \alpha |c_2|\big|}{\mu + 3} = \frac{\left|\frac{\beta(\mu+2)}{\alpha} - \alpha\right|}{\mu + 3} \cdot |c_2| \le \max\left(\frac{\beta}{\alpha} \cdot \frac{\mu + 2}{\mu + 3}, \frac{\alpha}{\mu + 3}\right) \cdot |c_2| \\ &\le |c_2|. \end{aligned}$$

For $k \ge 1$, using induction and (8), we derive

$$\begin{aligned} |c_{2k+2}| &\le \frac{\alpha |c_{2k+1}| + (\mu + 2k - \beta)|c_{2k}|}{\mu + 2k + 2} \le \frac{\alpha + \mu + 2k - \beta}{\mu + 2k + 2} \varphi_k |c_2| \\ &= \psi_{k+1} \varphi_k |c_2| = \varphi_{k+1} |c_2|. \end{aligned} \tag{13}$$

This is the left inequality of (12) for $k + 1$. For the right inequality we use (8), (13), and the induction hypothesis $|c_{2k+1}| \le \varphi_k |c_2|$ to obtain

$$\begin{aligned} |c_{2k+3}| &\le \frac{\alpha |c_{2k+2}| + (\mu + 2k + 1 - \beta)|c_{2k+1}|}{\mu + 2k + 3} \\ &\le \frac{\alpha \psi_{k+1} + \mu + 2k + 1 - \beta}{\mu + 2k + 3} \cdot \varphi_k |c_2|. \end{aligned}$$

Thus, since $\varphi_{k+1} = \psi_{k+1} \varphi_k$, it remains to show that

$$\frac{\alpha \psi_{k+1} + \mu + 2k + 1 - \beta}{\mu + 2k + 3} \le \psi_{k+1}. \tag{14}$$

By (9), $\mu + 2k + 3 - \alpha \ge 2k > 0$ and $2 - (\alpha - \beta) \ge 0$, so that (14) transforms to

$$\begin{aligned} & \frac{\mu + 2k + 1 - \beta}{\mu + 2k + 3 - \alpha} \le \psi_{k+1} \\ \Leftrightarrow \quad & 1 - \frac{2 - (\alpha - \beta)}{\mu + 2k + 3 - \alpha} \le \psi_{k+1} = 1 - \frac{2 - (\alpha - \beta)}{\mu + 2k + 2} \\ \Leftarrow \quad & \mu + 2k + 3 - \alpha \le \mu + 2k + 2 \\ \Leftrightarrow \quad & 1 \le \alpha. \end{aligned}$$

Since $\alpha \geq 1$ holds true by (9), this finishes the proof of (12).

The recurrence relation $\Gamma(z+1) = z\Gamma(z)$ of the gamma function gives

$$\prod_{k=0}^{m} \frac{x+k}{y+k} = \frac{\Gamma(y)\Gamma(x+m+1)}{\Gamma(x)\Gamma(y+m+1)} \qquad \text{for } x, y \in \mathbb{R}_{>0} \text{ and } m \in \mathbb{N} \tag{15}$$

because

$$\frac{\Gamma(x+m+1)}{\Gamma(y+m+1)} = \frac{\Gamma(x+m)}{\Gamma(y+m)} \cdot \frac{x+m}{y+m} = \cdots = \frac{\Gamma(x)}{\Gamma(y)} \prod_{k=0}^{m} \frac{x+k}{y+k}.$$

Abbreviate, for $m \geq 1$,

$$z_m := \max(|c_{2m}|, |c_{2m+1}|), \quad \hat{x} := (\mu + \alpha - \beta)/2 + 1, \quad \hat{y} := \mu/2 + 2. \tag{16}$$

Note that $z_1 = |c_2|$ by the right inequality in (12) for $k = 1$. Then $\psi_k = \frac{\hat{x}+k-2}{\hat{y}+k-2}$ for $k \geq 2$, and by (9) and $\mu \geq 0$ we have $\hat{x} \geq 1 > 0$ and $\hat{y} \geq 2 > 0$. Thus, (12) and (15) imply

$$z_m \leq \prod_{k=2}^{m} \frac{\hat{x}+k-2}{\hat{y}+k-2} \cdot z_1 = \prod_{k=0}^{m-2} \frac{\hat{x}+k}{\hat{y}+k} \cdot z_1 = \frac{\Gamma(\hat{y})\Gamma(\hat{x}+m-1)}{\Gamma(\hat{x})\Gamma(\hat{y}+m-1)} \cdot z_1. \tag{17}$$

Next, recall Gautschi's inequality for the gamma function ([1], 5.6.4, p. 138):

$$x^{1-r} \leq \frac{\Gamma(x+1)}{\Gamma(x+r)} \leq (x+1)^{1-r} \qquad \text{for } x \in \mathbb{R}_{>0} \text{ and } r \in [0,1] \; ^{1} \tag{18}$$

and also the remainder estimate for the Hurwitz zeta function $\zeta(s,q) := \sum_{k=0}^{\infty} (k+q)^{-s}$, see [1], 25.11.5, p. 608:

$$\begin{aligned} \sum_{k=N+1}^{\infty} \frac{1}{(k+q)^s} &= \zeta(s,q) - \sum_{k=0}^{N} \frac{1}{(k+q)^s} \\ &= \frac{(N+q)^{1-s}}{s-1} - s \int_N^{\infty} \frac{x - \lfloor x \rfloor}{(x+q)^{s+1}} \, dx \\ &\leq \frac{(N+q)^{1-s}}{s-1} \qquad \text{for } s > 1,\ q > 0,\ N \in \mathbb{N}_0 \end{aligned} \tag{19}$$

Since $r := \hat{x} - \hat{y} + 1 = \frac{\alpha-\beta}{2} \in [0,1]$, (17) and (18) with $x := \hat{x} - r = \hat{y} - 1 > 0$ and $x := \hat{y} + m - 2 > 0$, respectively, imply

$$\begin{aligned} z_m &\leq \frac{\Gamma(\hat{y})}{\Gamma(\hat{x})} \cdot \frac{\Gamma(\hat{x}+m-1)}{\Gamma(\hat{y}+m-1)} z_1 \leq \hat{y}^{1-r} (\hat{y} - 2 + m)^{r-1} z_1 \\ &= (\mu/2+2)^{1-\frac{\alpha-\beta}{2}} (\mu/2+m)^{\frac{\alpha-\beta}{2}-1} z_1. \end{aligned}$$

Using $z_1 = |c_2|$, set

$$\nu := \left((\mu/2+2)^{1-\frac{\alpha-\beta}{2}} z_1 \right)^2 = (\mu/2+2)^{2-(\alpha-\beta)} \frac{\alpha^2}{(\mu+1)^2(\mu+2)^2} \; . \tag{20}$$

[1] The inequality is strict for $r \in (0,1)$.

From now on, we need the original assumption (2). Then, using (19) with $s := 2 - (\alpha - \beta) > 1$, $q := \mu/2 > 0$, and $N := 0$, the Euclidean norm of $z = (z_m)_{m=1,\ldots,\lfloor n/2 \rfloor}$ is bounded by

$$\begin{aligned} \|z\|^2 &\leq \nu \cdot \sum_{m=1}^{\infty} (\mu/2 + m)^{\alpha-\beta-2} \leq \nu \cdot \frac{(\mu/2)^{\alpha-\beta-1}}{1-\alpha+\beta} \\ &= \frac{\alpha^2}{(1-\alpha+\beta)} \cdot \frac{(\mu+4)(1+4/\mu)^{1-\alpha+\beta}}{(\mu+2)^2} \cdot \frac{1}{2(\mu+1)^2} \\ &= \frac{\theta(\mu)}{2(\mu+1)^2} \end{aligned} \tag{21}$$

where $\theta(\mu)$ is defined by (3). For later use, we note that $\theta(\mu)$ is monotonically decreasing. This is because both functions $\frac{(\mu+4)}{(\mu+2)^2}$ and $(1+4/\mu)^{1-\alpha+\beta}$ decrease in μ, where $1 - \alpha + \beta > 0$ is used. Combining (5), (16), and (21) supplies

$$\|c\|^2 \leq c_1^2 + 2\|z\|^2 \leq \frac{1+\theta(\mu)}{(\mu+1)^2}. \tag{22}$$

For $j \in \{1, \ldots, n\}$ let $\widehat{A}$ denote the lower right submatrix of A of order $n - j + 1$. Then $\widehat{A}$ has the same pattern as A with $\hat{\mu} := \mu + j$ instead of μ. Since A is a lower triangular matrix, $\widehat{A}^{-1}$ is the lower right submatrix of A^{-1}. Thus, the norm of the first column $\hat{c}$ of $\widehat{A}^{-1}$ is that of the j-th column of A^{-1}. From (22) and since $\theta(\mu)$ is decreasing it follows that

$$\|\hat{c}\|^2 \leq \frac{1+\theta(\mu+j)}{(\mu+j+1)^2} \leq \frac{1+\theta(\mu)}{(\mu+j+1)^2}. \tag{23}$$

Thus, using also (19) with $s := 2$, $q := \mu + 1$, and $N := 0$, the Frobenius norm of A^{-1} is estimated by

$$\|A^{-1}\|_F^2 \leq (1+\theta(\mu)) \sum_{j=1}^{\infty} \frac{1}{(\mu+j+1)^2} \leq \frac{1+\theta(\mu)}{\mu+1} \ . \tag{24}$$

Therefore the smallest singular value σ_n of A is bounded from below by

$$\sigma_n \quad \geq \quad \sqrt{\frac{\mu+1}{1+\theta(\mu)}} =: \omega \tag{25}$$

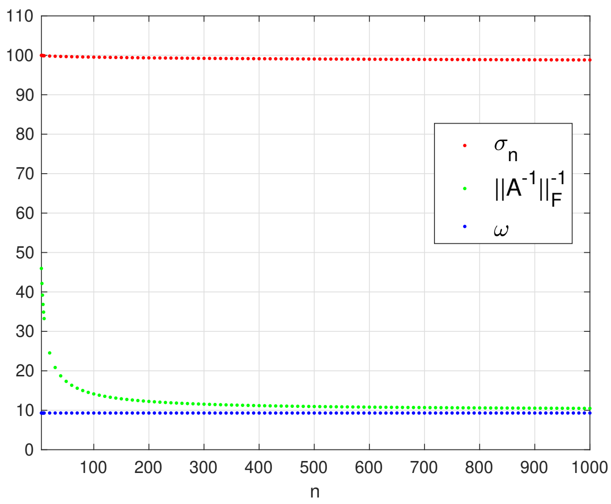
and this positive lower bound ω does not depend on the dimension n. □

Example 2. The original values in Watanabe's problem are $\alpha := \frac{7}{3}$, $\beta := \frac{5}{3}$, $\mu := 100 - \frac{1}{6}$, and they fulfill (2).

For Watanabe's problem we obtain $\omega = 9.300556...$. The following MATLAB program computes for varying dimension n the smallest singular value σ_n of A as well as $\|A^{-1}\|_F^{-1}$ and plots the result. The figure shows that ω is close to the asymptotically sharp lower bound of $\|A^{-1}\|_F^{-1}$ which seems to be about 10.

The asymptotically sharp lower bound for σ_n seems to be about 98 which is a factor 10.5 larger than ω.

```
alpha = 7/3;
beta = 5/3;
mu = 100-1/6;
theta = alpha^2*mu*(1+4/mu)^(2-alpha+beta)/((1-alpha+beta)*(mu+2)^2);
omega = sqrt((mu+1)/(1+theta));
N = [5:9,10:10:1000];
sigma_n = zeros(size(N));
f = zeros(size(N));
for i = 1:length(N)
    n = N(i);
    col = zeros(n,1);
    col(2:2:end) = alpha;
    col(3:2:end) = beta;
    A = toeplitz(col,zeros(1,n)) + diag(mu+(1:n));
    B = inv(A);
    sigma_n(i) = min(svd(A));
    f(i) = 1/norm(B,'fro');
end
plot(N,sigma_n,'.r',N,f,'.g',N,omega*ones(size(N)),'.b');
legend({'\sigma_n','||A^{-1}||_F^{-1}','\omega'},'FontSize',14)
xlabel('n'); axis([min(N),max(N),0,110]); grid on
```



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