POSITIVE ENTRIES OF STABLE MATRICES[*]

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Abstract. The question of how many elements of a real positive stable matrix must be positive is investigated. It is shown that any real stable matrix of order greater than 1 has at least two positive entries. Furthermore, for every stable spectrum of cardinality greater than 1 there exists a real matrix with that spectrum with exactly two positive elements, where all other elements of the matrix can be chosen to be negative.

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1. Introduction. For a square complex matrix A let $\sigma(A)$ be the spectrum of A, that is, the set of eigenvalues of A listed with their multiplicities. Recall that a (multi) set of complex numbers is called *(positive) stable* if all the elements of the set have positive real parts, and that a square complex matrix A is called *stable* if $\sigma(A)$ is stable. In this paper we investigate the question of how many elements of a real stable matrix must be positive.

We first show that a stable real matrix A has either positive diagonal elements or it has at least one positive diagonal element and one positive off-diagonal element. We then show that for any stable n-tuple ζ of complex numbers, $n > 1$, such that ζ is symmetric with respect to the real axis, there exists a real stable $n \times n$ matrix A with exactly two positive entries such that $\sigma(A) = \zeta$.

The stable $n \times n$ matrix with exactly two positive entries, whose existence is proven in Section 2, has $(n-1)^2$ zeros in it. In Section 3 we prove that for any stable n-tuple ζ of complex numbers, $n > 1$, such that ζ is symmetric with respect to the real axis, there exists a real stable $n \times n$ matrix A with two positive entries and all other entries negative such that $\sigma(A) = \zeta$.

In Section 4 we suggest some alternative approaches to obtain the results of Section 2.

2. Positive entries of stable matrices. Our aim in this section is to show that for any stable n-tuple ζ of complex numbers, $n > 1$, consisting of real numbers and conjugate pairs, there exists a real stable $n \times n$ matrix A with exactly two positive

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entries such that $\sigma(A) = \zeta$. We shall first show that every real stable matrix of order greater than 1 has at least two positive elements. In fact we show more than that, that is, that for a stable real matrix A either all diagonal elements of A are positive or A must have at least one positive entry on the main diagonal and one off the main diagonal.

NOTATION 2.1. For an n-tuple $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ of complex numbers we denote by $s_1(\zeta), \ldots, s_n(\zeta)$ the elementary symmetric functions of ζ, that is,

$$s_k(\zeta) = \sum_{1 \le i_1 < \ldots < i_k \le n} \zeta_{i_1} \cdot \ldots \cdot \zeta_{i_k}, \qquad k = 1, \ldots, n.$$

Also, we let $s_0(\zeta) = 1$ and $s_k(\zeta) = 0$ whenever $k > n$ or $k < 0$. We say that ζ has positive elementary symmetric functions whenever $s_k(\zeta) > 0, \ \ k = 1, \ldots, n$.

LEMMA 2.2. *Let $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ be an n-tuple of complex numbers with positive elementary symmetric functions. Then ζ contains no nonpositive real numbers.*

Proof. Note that ζ has positive elementary symmetric functions if and only if the polynomial $p(x) = \prod_{i=1}^{n}(x + \zeta_i)$ has positive coefficients. It follows that $p(x)$ cannot have nonnegative roots, implying that none of the ζ_i's is a nonpositive real number. □

NOTATION 2.3. For $\mathbb{F} = \mathbb{R}, \mathbb{C}$, the fields of real and complex numbers respectively, we denote by $\mathrm{M_n}(\mathbb{F})$ the algebra of $n \times n$ matrices with entries in $\mathbb{F}$. For $A = (a_{ij})_1^n \in \mathrm{M_n}(\mathbb{F})$ we denote by $\operatorname{tr} A$ the trace of A, that is, the sum $\sum_{i=1}^{n} a_{ii}$.

PROPOSITION 2.4. *Let $A = (a_{ij})_1^n \in \mathrm{M_n}(\mathbb{R})$, and assume that $\sigma(A)$ has positive elementary symmetric functions. Then either all the diagonal elements of A are positive or A has at least one positive diagonal element and one positive off-diagonal element.*

Proof. As is well known, the trace of A is equal to $s_1(\sigma(A))$, and so we have $\sum_{i=1}^{n} a_{ii} > 0$, and it follows that at least one diagonal element of A is positive. Assume that that all off-diagonal elements of A are nonpositive. Such a real matrix is called a *Z-matrix*. Since the elementary symmetric functions of $\sigma(A)$ are positive, it follows by Lemma 2.2 that A has no nonpositive real eigenvalues. Since a Z-matrix has no nonpositive real eigenvalues if and only if all its principal minors are positive, e.g. [1, Theorem (6.2.3), page 134], it follows that all the diagonal elements of A are positive. □

NOTATION 2.5. For an n-tuple $\zeta = \{\zeta_1, \ldots, \zeta_n\}$ of complex numbers we denote by $\overline{\zeta}$ be the n-tuple $\{\overline{\zeta}_1, \ldots, \overline{\zeta}_n\}$. We say that $\overline{\zeta} = \zeta$ whenever the two sets $\overline{\zeta}$ and ζ are identical.

Note that $\overline{\zeta} = \zeta$ if and only if all elementary symmetric functions of ζ are real.

The following result is well known, and we provide a proof for the sake of completeness.

PROPOSITION 2.6. *Let ζ be a stable n-tuple of complex numbers such that $\overline{\zeta} = \zeta$. Then ζ has positive elementary symmetric functions.*

Proof. We prove our claim by induction on n. For $n = 1, 2$ the result is trivial. Assume that the result holds for $n \leq m$ where $m \geq 2$, and let $n = m + 1$. Assume first that ζ contains a positive number λ, and let ζ' be the $(n-1)$-tuple obtained by eliminating λ from ζ. Note that ζ' is stable and $\overline{\zeta'} = \zeta'$. By the inductive assumption we have $s_k(\zeta') > 0$, $k = 1, \ldots, n-1$, and it follows that

$$s_k(\zeta) = s_k(\zeta') + \lambda s_{k-1}(\zeta') > 0, \quad k = 1, \ldots, n.$$

If ζ does not contain a positive number then it contains a conjugate pair $\{\lambda, \overline{\lambda}\}$, where $\mathrm{Re}(\lambda) > 0$. Let ζ'' be the $(n-2)$-tuple obtained by eliminating λ and $\overline{\lambda}$ from ζ. Note that the ζ'' is stable and $\overline{\zeta''} = \zeta''$. By the inductive assumption we have $s_k(\zeta'') > 0$, $k = 1, \ldots, n-2$, and it follows that

$$s_k(\zeta) = s_k(\zeta'') + 2\mathrm{Re}(\lambda) s_{k-1}(\zeta'') > 0 + |\lambda|^2 s_{k-2}(\zeta'') > 0, \quad k = 1, \ldots, n.$$

proving our claim. ∎

It is easy to show that the converse of Proposition 2.6 holds when $n \leq 2$. However, the converse does not hold for a larger n, as is demonstrated by the nonstable triple $\zeta = \{\ 3\ ,\ -1 + 3i\ ,\ -1 - 3i\ \}$, whose elementary symmetric functions are positive.

As a corollary of Propositions 2.4 and 2.6 we obtain

Corollary 2.7. *Let A be a stable real square matrix. Then either all the diagonal elements of A are positive or A has at least one positive diagonal element and one positive off-diagonal element.*

In order to prove the existence of a real stable $n \times n$ matrix A with exactly two positive entries, we introduce:

Notation 2.8. Let n be a positive integer. For an n-tuple ζ of complex numbers we denote by $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ the matrices

$$C_1(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-1} s_n(\zeta) \\ 1 & 0 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-2} s_{n-1}(\zeta) \\ 0 & 1 & 0 & \ldots & 0 & 0 & 0 & (-1)^{n-3} s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 & 0 & -s_2(\zeta) \\ 0 & 0 & 0 & \ldots & 0 & 0 & 1 & s_1(\zeta) \end{pmatrix},$$

$$C_2(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 0 & 0 & s_n(\zeta) \\ -1 & 0 & 0 & \ldots & 0 & 0 & 0 & s_{n-1}(\zeta) \\ 0 & -1 & 0 & \ldots & 0 & 0 & 0 & s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 & 0 & s_2(\zeta) \\ 0 & 0 & 0 & \ldots & 0 & 0 & -1 & s_1(\zeta) \end{pmatrix},$$

$$C_3(\zeta) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0 & -s_n(\zeta) \\ -1 & 0 & 0 & \dots & 0 & 0 & 0 & -s_{n-1}(\zeta) \\ 0 & -1 & 0 & \dots & 0 & 0 & 0 & -s_{n-2}(\zeta) \\ \vdots & \vdots & \vdots & \vdots\vdots\vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -1 & 0 & -s_2(\zeta) \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & s_1(\zeta) \end{pmatrix}.$$

Recall that $A \in \mathrm{M_n}(\mathbb{C})$ is called nonderogatory if for every eigenvalue λ of A the Jordan canonical form of A has exactly one Jordan block corresponding to λ. Equivalently, the minimal polynomial of A is equal to the characteristic polynomial of A.

LEMMA 2.9. *Let n be a positive integer, $n > 1$, and let $\zeta = \{\zeta_1, \dots, \zeta_n\}$ be an n-tuple of complex numbers. Then the matrices $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ are diagonally similar, are nonderogatory and share the spectrum ζ.*

Proof. The matrix $C_1(\zeta)$ is the companion matrix of the polynomial $q(x) = \prod_{i=1}^n (x - \zeta_i)$. Hence $\sigma(C_1(\zeta)) = \zeta$ and $C_1(\zeta)$ is nonderogatory. Clearly

$$C_2(\zeta) = D_1 C_1(\zeta) D_1, \quad \text{where } D_1 = \mathrm{diag}((-1)^1, (-1)^2, \dots, (-1)^n),$$

and

$$C_3(\zeta) = D_2 C_3(\zeta) D_2, \quad \text{where } D_2 = \mathrm{diag}(1, 1, \dots, 1, -1).$$

Our claim follows. □

In view of Lemma 2.9, the claim of Proposition 2.6 on $C_3(\zeta)$ yields the following main result of this section.

THEOREM 2.1. *Let n be a positive integer, $n > 1$, and let ζ be an n-tuple of complex numbers such that $\overline{\zeta} = \zeta$. If ζ has positive elementary symmetric functions then there exists a matrix $A \in \mathrm{M_n}(\mathbb{R})$ such that $\sigma(A) = \zeta$ and A has one positive diagonal entry and one positive off-diagonal entry, while all other entries of A are nonpositive. In particular, every nonderogatory stable matrix $A \in \mathrm{M_n}(\mathbb{R})$ is similar to a real $n \times n$ matrix which has exactly two positive entries.*

3. Eliminating the zero entries. The proof of Theorem 2.1 uses the matrix $C_3(\zeta)$ which has $(n-1)^2$ zero entries. The aim of this section is to strengthen Theorem 2.1 by replacing $C_3(\zeta)$ with a real matrix A, having exactly two positive entries, all other entries being negative and $\sigma(A) = \zeta$.

We start with a weaker result, which one gets easily using perturbation techniques. Let $A \in \mathrm{M_n}(\mathbb{R})$ and let $\|\cdot\| : \mathrm{M_n}(\mathbb{R}) \to [0, \infty)$ be the l_2 operator norm. Since the eigenvalues of a A depend continuously on the entries of the A, it follows that if $\sigma(A)$ has positive elementary symmetric functions, then for $\varepsilon > 0$ sufficiently small, every matrix $\tilde{A} \in \mathrm{M_n}(\mathbb{R})$ with $\|\tilde{A} - A\| < \varepsilon$ has a spectrum $\sigma(\tilde{A})$ with positive elementary symmetric functions. Also, if A is stable then for $\varepsilon > 0$ sufficiently small, every matrix

$\tilde{A} \in \mathrm{M_n}(\mathbb{R})$ with $\|\tilde{A} - A\| < \varepsilon$ is stable. Consequently, it follows immediately from Theorem 2.1 that

COROLLARY 3.1. *For a positive integer n, $n > 1$, there exists a matrix $A \in \mathrm{M_n}(\mathbb{R})$ such that $\sigma(A)$ has positive elementary symmetric functions and A has one positive diagonal entry and one positive off-diagonal entry, while all other entries of A are negative. Furthermore, the matrix A can be chosen to be stable.*

In the rest of this sectio we prove that one can find such a matrix A with any prescribed stable spectrum.

LEMMA 3.2. *Let n be a positive integer, $n > 1$, let ζ be an n-tuple of complex numbers such that $\overline{\zeta} = \zeta$, and assume that ζ has positive elementary symmetric functions. Suppose that there exists $X \in \mathrm{M_n}(\mathbb{R})$ such that*

$$(C_3(\zeta))_{ij} = 0 \implies (C_3(\zeta)X - XC_3(\zeta))_{ij} < 0, \qquad i, j = 1, \ldots, n.$$

Then there exist $A \in \mathrm{M_n}(\mathbb{R})$ similar to $C_3(\zeta)$ such that a_{nn}, $a_{n,n-1} > 0$ and all other entries of A are negative.

Proof. Assume the existence of such a matrix X. Define the matrix $T(t) = I - tX$, $t \in \mathbb{R}$. Let $r = ||X||^{-1}$. Using the Neumann series expansion, e.g. [2, page 7], for $|t| < r$ we have $T(t)^{-1} = \sum_{i=0}^{\infty} t^i X^i$. The matrix $A(t) = T(t)C_3(\zeta)T(t)^{-1}$ thus satisfies

$$A(t) = C_3(\zeta) + t(C_3(\zeta)X - XC_3(\zeta)) + O(t^2).$$

Therefore, there exists $\varepsilon \in (0, r)$ such that for $t \in (0, \varepsilon)$ the matrix $A(t)$ has positive entries in the $(n, n-1)$ and (n, n) positions, while all other entries of $A(t)$ are negative. □

The following lemma is well known, and we provide a proof for the sake of completeness.

LEMMA 3.3. *Let $A, B \in \mathrm{M_n}(\mathbb{F})$ where $\mathbb{F}$ is $\mathbb{R}$ or $\mathbb{C}$. The following are equivalent.*
(i) *The system $AX - XA = B$ is solvable over $\mathbb{F}$.*
(ii) *For every matrix $E \in \mathrm{M_n}(\mathbb{F})$ that commutes with A we have $\operatorname{tr} BE = 0$.*

Proof. (i)$\Longrightarrow$(ii). Let $E \in \mathrm{M_n}(\mathbb{F})$ commute with A. Then

$$\operatorname{tr} BE = \operatorname{tr}(AX - XA)E = \operatorname{tr} AXE - trXEA = \operatorname{tr} XEA - trXEA = 0.$$

(ii)$\Longrightarrow$(i). Consider the linear operator $L : \mathrm{M_n}(\mathbb{F}) \to \mathrm{M_n}(\mathbb{F})$ defined by $L(X) = AX - XA$. Its kernel consists of all matrices in $\mathrm{M_n}(\mathbb{F})$ commuting with A. By the previous implication, the image of L is contained in the subspace V of $\mathrm{M_n}(\mathbb{F})$ consisting of all matrices C such that $\operatorname{tr} CE = 0$ whenever $E \in \mathrm{kernel}(L)$. Since clearly $\dim(V) = n^2 - \dim(\mathrm{kernel}(L)) = \dim(\mathrm{image}(L))$, it follows that $\mathrm{image}(L) = V$. □

THEOREM 3.1. *Let n be a positive integer, $n > 1$, and let ζ be an n-tuple of complex numbers. Let b_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ be given complex numbers,* *and let $C = C_k(\zeta)$ for some $k \in \{1, 2, 3\}$. Then there exists unique $b_{in} \in \mathbb{C}$, $i = 1, \ldots, n$, such that for the matrix $B = (b_{ij})_1^n \in \mathrm{M_n}(\mathbb{C})$ the system $CX - XC = B$ is solvable. Furthermore, if $\overline{\zeta} = \zeta$ and b_{ij} is real for $i = 1, \ldots, n$, $j = 1, \ldots, n-1$, then the matrix B is real, and the solution X can be chosen to be real.*

Proof. Since $C_2(\zeta)$ and $C_3(\zeta)$ are diagonally similar to $C_1(\zeta)$, where the corresponding diagonal matrices are real, it is enough to prove the theorem for $C = C_1(\zeta)$. So, let $C = C_1(\zeta)$ and consider the system

$$CX - XC = B. \tag{3.3.1}$$

As is well known, e.g. [3, Corollary 1, page 222], since $C = C_1(\zeta)$ is nonderogatory, every matrix that commutes with C is a polynomial in C. Therefore, it follows from Lemma 3.3 that the system (3.3.1) is solvable if and only if

$$\operatorname{tr} BC^k = 0, \quad k = 0, \ldots, n-1. \tag{3.3.2}$$

Denote $v_k = b_{n+1-k,n}$, $k = 1, \ldots, n$. Note that (3.3.2) is a system of n equations in the variables $v_1, \ldots, v_n$. Furthermore, it is easy to verify that the first nonzero element in the nth row of C^k is located at the position $(n, n-k)$ and its value is 1. It follows that if we write (3.3.2) as $Ev = f$, where $E \in \mathrm{M_n}(\mathbb{C})$ and $v = (v_1, \ldots, n)^T$, then E is a lower triangular matrix with 1's along the main diagonal. It follows that the matrix B is uniquely determined by (3.3.2).

If $\overline{\zeta} = \zeta$ and b_{ij} is real for $i = 1, \ldots, n$, $j = 1, \ldots, n-1$ then $C = C_1(\zeta)$ is real and hence the system (3.3.2) has real coefficients, and the uniquely determined B is real. It follows that the system (3.3.1) is real, and so it has a real solution X. □

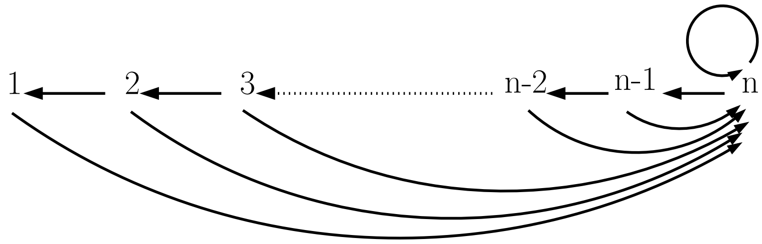
If we choose the numbers b_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, n-1$, to be negative, then Lemma 3.2 and Theorem 3.1 yield

COROLLARY 3.4. *Let n be a positive integer, $n > 1$, let ζ be an n-tuple of complex numbers, and assume that the elementary symmetric functions of ζ are positive. Then there exists a matrix $A \in \mathrm{M_n}(\mathbb{R})$ with $\sigma(A) = \zeta$ such that A has one positive diagonal element, one positive off-diagonal element and all other entries of A are negative. In particular, the above holds for stable n-tuples ζ such that $\overline{\zeta} = \zeta$.*

4. Other types of companion matrices. Another way to prove some of the results of Section 2 is to parameterize the companion matrices in Notation 2.8. Consider

$$C = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_0 \\ \beta_0 & 0 & 0 & \cdots & 0 & 0 & 0 & \gamma_1 \\ 0 & \beta_1 & 0 & \cdots & 0 & 0 & 0 & \gamma_2 \\ & & & \cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & \beta_{n-3} & 0 & \gamma_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 0 & \beta_{n-2} & \gamma_{n-1} \end{pmatrix}$$

From looking at the directed graph of C which is



one can immediately see that there is exactly one simple cycle of length k for $1 \leq k \leq n$, that is, $(n, \ldots, n+1-k)$. Therefore, the only nonzero principal minors of C are those whose rows and columns are indexed by $\{k, \ldots, n\}$, $k = 1, \ldots, n$, and their respective values are $(-1)^{n-k}\gamma_{k-1}\beta_{k-1}\cdots\beta_{n-2}$ for $k < n$ and γ_{n-1} for $k = n$. It follows that the characteristic polynomial $\chi_C(x)$ of C is

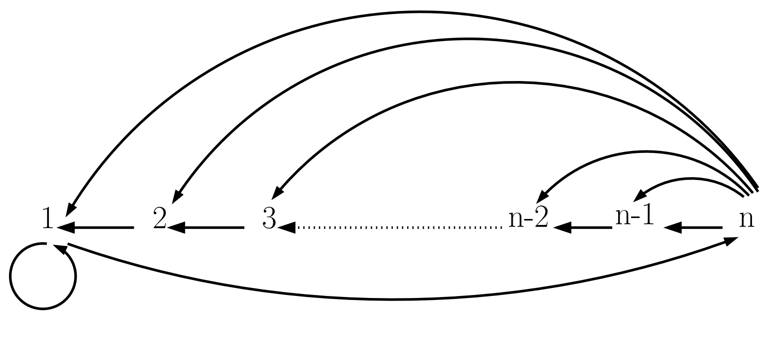
$$\begin{aligned} \chi_C(x) \quad = \quad & x^n - \gamma_{n-1}x^{n-1} - \gamma_{n-2}\beta_{n-2}x^{n-2} - \gamma_{n-3}\beta_{n-3}\beta_{n-2}x^{n-3} - \ldots \\ & \ldots - \gamma_1\beta_1\beta_2\cdots\beta_{n-2}x - \gamma_0\beta_0\beta_1\cdots\beta_{n-2}. \end{aligned} \tag{4.4.1}$$

Using this explicit formula, one can prove directly the claim contained in Lemma 2.9 that the matrices $C_1(\zeta)$, $C_2(\zeta)$ and $C_3(\zeta)$ share the spectrum ζ.

There are other possibilities to generate companion matrices. For example, consider the matrix

$$L = \begin{pmatrix} \gamma_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & \beta_{n-2} \\ \beta_{n-3} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \beta_{n-4} & 0\cdots & 0 & 0 & 0 & 0 & \\ & & & \cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & \beta_0 & 0 & 0 \\ \gamma_{n-2} & \gamma_{n-3} & \gamma_{n-4} & \cdots & \gamma_2 & \gamma_1 & \gamma_0 & 0 \end{pmatrix}.$$

The directed graph of L is



Again it is clear that there is exactly one simple cycle of length k for any $1 \leq k \leq n$, that is, (1) for $k = 1$ and $(n, k-1, \ldots, 1)$ for $1 < k \leq n$. Therefore, the only nonzero 1×1 principal minor of L is $l_{11} = \gamma_{n-1}$, and for $1 < k \leq n$ the only nonzero $k \times k$ principal minor of L is the one whose rows and columns are indexed by $\{1, \dots, k-1, n\}$, and its value is $(-1)^{n-k}\gamma_{k-1}\beta_{k-1}\cdots\beta_{n-2}$. It follows that the characteristic polynomial $\chi_L(x)$ of L is identical to $\chi_C(x)$. Note that there is no permutation matrix P with $P^TCP = L$ or $P^TC^TP = L$.

Now, take the following specific choice of the parameters β and γ

$$L_1 = \begin{pmatrix} -p_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & \cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 \\ -p_{n-2} & -p_{n-3} & -p_{n-4} & \cdots & -p_2 & -p_1 & -p_0 & 0 \end{pmatrix}.$$

By (4.4.1), the characteristic polynomial computes to

$$\chi_{L_1}(x) = \sum_{\nu=0}^{n} p_\nu x^\nu,$$

where $p_n = 1$.

So L_1 is another kind of companion matrix. Note that L_1 is almost lower triangular, with only one nonzero element above the main diagonal and one on the main diagonal.

Another specific choice of the parameters β and γ can be used to produce another direct proof of Theorem 2.1. For an n-tuple ζ of complex numbers with $\zeta = \overline{\zeta}$ and positive elementary symmetric functions, the polynomial $q(x) = \prod_{i=1}^{n}(x - \zeta_i) = \sum_{i=0}^{n} q_i x^i$ has coefficients q_i, $0 \le i \le n$ of alternating signs, where $q_n = 1$. By (4.4.1), the polynomial $q(x)$ is the characteristic polynomial of the matrix

$$\begin{pmatrix} -q_{n-1} & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ & & & \cdots\cdots\cdots & & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 0 \\ -q_{n-2} & q_{n-3} & -q_{n-4} & \cdots & (-1)^{n-3}q_2 & (-1)^{n-2}q_1 & (-1)^{n-1}q_0 & 0 \end{pmatrix}.$$

which has exactly two positive entries, that is, $-q_{n-1}$ on the diagonal and 1 in the right upper corner.

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