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On the generation of very ill-conditioned integer matrices

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Abstract: In this paper we study the generation of an ill-conditioned integer matrix $A = [a_{ij}]$ with $|a_{ij}| \leq \mu$ for some given constant μ. Let n be the order of A. We first give some upper bounds of the condition number of A in terms of n and μ. We next propose new methods to generate extremely ill-conditioned integer matrices. These methods are superior to the well-known method by Rump in some respects, namely, the former has a simple algorithm to generate a larger variety of ill-conditioned matrices. In particular we propose a method to generate ill-conditioned matrices with a choice of desirable *singular value distributions* as benchmark matrices.

Key Words: ill-conditioned matrix, condition number, verified numerical computation, integer matrix, singular value distribution, companion matrix

1. Introduction

Extremely ill-conditioned matrices are required to examine the quality of verified numerical computations for solving simultaneous linear equations [1–6]. Here an ill-conditioned matrix means that its condition number $\mathrm{Cond}(A) = \|A\|\|A^{-1}\|$ is 10^{16} or larger in IEEE 754 binary64 (double precision) arithmetic. Though the condition number of a matrix is the most important index in numerical analysis, its properties such as upper bounds for integer matrices discussed in this paper have not been fully investigated [1, 12–14].

Once S. Rump [7] proposed a method to generate extremely ill-conditioned matrices with floating-point entries. The method is now most well-known and a modification is used as the standard tool to generate an ill-conditioned matrix in INTLAB (see "*randmat*"). However, since his method is based on *Pell's equation* (which is well-known in number theory) with a limited number of solutions, there is not so much variety of matrices to be generated. Therefore we are seeking alternative methods to obtain a greater variety of ill-conditioned matrices. Moreover we are interested in *a priori* specifiable singular value distributions. Up to now several methods [8–11] which may be considered as extensions of Rump's method [7] were proposed. In this paper we give new generation methods of *ill-conditioned*

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integer matrices[1].

Section 2 introduces to the subject and summarizes related previous results. Throughout the paper μ denotes a large positive integer such as 10^8, 10^{16} or 2^{53} (but $\mu = 10$ or less may also, theoretically, be permissible). We are aiming to generate extremely ill-conditioned $n \times n$ integer matrices $A = [a_{ij}]$ with $|a_{ij}| \le \mu$.

In Section 3 several upper bounds of the 2-norm condition number are shown in terms of n and μ, in particular for specific distributions of the singular values of A. The results in Section 3 are closely related to Section 5. In Section 4 we give a new generation method[2] for extremely ill-conditioned matrices. The method may be regarded as a modification of Rump's method [7] and has the following features in comparison with Rump's original method [7] and its extensions [8–10]: (i) the algorithm is simpler, (ii) the obtainable condition number is roughly the same as for previous methods, and (iii) it can generate a greater variety of matrices. The obtained matrices are somewhat similar to a companion matrix. Some numerical examples are shown for illustration.

The matrices generated up to Section 4 in this paper as well as those by previous methods [7–11], in particular by Rump's method, bear the drawback that the first $n-1$ singular values are large and not far from the first one, the spectral norm, whereas only the n-th singular value is extremely small. This implies that the inverse of those matrices is very near to a matrix *with rank one* (see examples in Section 4). This may not necessarily be preferable for a benchmark matrix. Therefore we consider in Section 5 some *desirable* distributions of singular values of a matrix and give partial solutions to the generation problem by using some special types of matrices. We will see that the condition number of the obtained matrices is nearly optimal in some sense. Numerical examples show good agreement with theory.

2. Preliminaries

The main purpose of this paper is to generate an $n \times n$ **integer** matrix $A = [a_{ij}]$ which satisfies

$$|a_{ij}| \le \mu \quad (i, j = 1, 2, \cdots, n) \tag{1}$$

and whose condition number $\mathrm{Cond}(A) = \|A\|\|A^{-1}\|$ is extremely large[3]. It is obvious that the maximally achievable condition number of A increases monotonically with μ. In this paper we assume[4]

$$|\det(A)| = 1. \tag{2}$$

Note that Eq. (2) can be achieved only under very delicate relations among the entries a_{ij}, in particular when n and μ are large.

We will briefly summarize previous results.

2.1 Rump's method[7]

One of the key points of Rump's method is to find a 2×2 integer matrix V s.t.

$$V = \begin{bmatrix} P & kQ \\ Q & P \end{bmatrix}, \ \det(V) = \det\left(\begin{bmatrix} P & kQ \\ Q & P \end{bmatrix}\right) = 1, \tag{3}$$

where k is a prescribed small positive integer[5]. The integers P and Q are very large (depending on the desired condition number), such as 10^{50}, and are chosen to satisfy *Pell's equation*

[1] Though our final matrices are to be **floating-point** matrices, we discuss about the generation of **integer** matrices due to the same reason as in [7].

[2] The matrices generated in this paper as well as in the previous ones [7–9] are very sparse. For example, there are few elements equal to ± 1, and the number of nonzero elements not equal to ± 1 is only about cn, where $c = 2$ or $c = 3$. For practical use we can derive dense matrices by multiplying them by appropriate matrices P with $|\det(P)| = 1$ from left and/or right. We can also add rows or columns by similar operations. Note, however, that this may change the condition number.

[3] The infinity norm $\|A\|_\infty$ defined by $\max_i \sum_{j=1}^n |a_{ij}|$ is often used in verified numerical analysis, but the spectral norm $\|A\|_2$ is also used in this paper. Similarly, both $\mathrm{Cond}_\infty(A)$ and $\mathrm{Cond}_2(A)$ are used. See also Eqs. (10) and (11).

[4] This is not a strict condition, but proved to be important to generate ill-conditioned integer matrices.

[5] The symbols n and V in Eqs. (1) and (3) are different from those in the Rump's paper [7]. The order "$2(n+1)$" of the matrix in [7] is written as "n" in this paper.

$$P^2 - kQ^2 = 1, \tag{4}$$

from which $\det(V) = 1$ in Eq. (3) follows. Utilizing V in Eq. (3), Rump constructed an $n \times n$ (n is even) integer matrix A in Eq. (1.6) of [7] and showed by rather tricky calculations that

$$\mathrm{Cond}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty \geq (P + kQ)^2. \tag{5}$$

Under the reasonable assumption that P, $Q \sim \mu^{n/2}$ we have

$$\mathrm{Cond}_\infty(A) \geq (P + kQ)^2 \sim 4\mu^n. \tag{6}$$

This shows that $\mathrm{Cond}_2(A)$ can be extremely large in Rump's method.

2.2 Previous extensions of Rump's method

Rump's algorithm can be generalized by replacing V in Eq. (3) with the following two kinds of matrices [8–10].

2.2.1 Replacing V by more general type of a 2×2 matrix

The matrix V in Eq. (3) was generalized into

$$V' = \begin{bmatrix} P & F \\ Q & G \end{bmatrix}, \quad \det(V') = PG - QF = 1. \tag{7}$$

For prescribed large integers P and Q having no common factor, e.g.,

$$\left.\begin{array}{rcl} P &=& 2^k, \;\; Q = 3^m \\ P &=& 2^{k_1} 5^{k_2} 11^{k_3}, \;\; Q = 3^{m_1} 7^{m_2} \end{array}\right\}, \tag{8}$$

we can find F and G satisfying Eq. (7) by using the extended Euclidean algorithm. The equations corresponding to Eqs. (5) and (6) also hold for this case. This method greatly enlarges the class of generated matrices.

2.2.2 Replacing V by a 3×3 matrix

The discussion similar to Eqs. (3) and (7) is possible for this case.

3. Upper bounds of condition number of matrices in terms of μ

In this section we show some upper bounds of $\mathrm{Cond}_2(A)$, the condition number of a matrix A with respect to *the spectral norm*[6], in terms of μ. As before, the general assumptions for this section are

$$|\det(A)| = 1 \quad \text{and} \quad |a_{ij}| \leq \mu \text{ for } 1 \leq i, j \leq n. \tag{9}$$

As we see, the condition numbers with *infinity norm* and the spectral norm are related by [12, 13]:

$$\frac{1}{\sqrt{n}}\|A\|_2 \leq \|A\|_\infty \leq \sqrt{n}\|A\|_2 \tag{10}$$

$$\therefore \quad \frac{1}{n}\mathrm{Cond}_2(A) \leq \mathrm{Cond}_\infty(A) \leq n\mathrm{Cond}_2(A). \tag{11}$$

This means that concerning large condition numbers both norms give almost the same value. So in this paper we use both norms conveniently; we use mainly the spectral norm in Sections 3 and 5 and the infinity norm in Section 4. Though our aim is to generate ill-conditioned *integer* matrices, we discuss on real matrices in most of Sections 3 and 5.

[6]The spectral norm is the matrix norm induced by the Euclidean vector norm, i.e. $\|A\|_2 := \max_{\|x\|=1} \|Ax\|$. It follows $\|A\|_2 = \sqrt{\varrho(A^T A)}$, where ϱ denotes the spectral radius.

3.1 Condition number of A in terms of μ for special singular value distributions

Let the eigenvalues of AA^T be $\lambda_i > 0$ $(i = 1, 2, \cdots, n)$, so that $\sigma_i \equiv \sqrt{\lambda_i}$ $(i = 1, 2, \cdots, n)$ are the singular values of A. We use decreasing order, i.e.

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n (> 0), \quad (\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \cdots \geq \sigma_{n-1} \geq \sigma_n > 0) \tag{12}$$

Here λ_i are the solutions of the characteristic equation

$$\det(\lambda \mathbf{1} - AA^T) = \lambda^n - \mathrm{tr}(AA^T)\lambda^{n-1} + \cdots + (-1)^n \det(AA^T) = 0, \tag{13}$$

where "tr(X)" denotes the trace of a matrix X and "$\mathbf{1}$" denotes the identity matrix. Note that $\mathrm{tr}(AA^T)$ is equal to the Frobenius norm $\|A\|_F^2$. Using $|a_{ij}| \leq \mu$ it follows

$$\sum_{i=1}^{n} \lambda_i = \|A\|_F^2 = \mathrm{tr}(AA^T) = \sum_{i,j=1}^{n} a_{ij}^2 \leq n^2 \mu^2, \tag{14}$$

and therefore

$$\|A\|_2 = \sigma_1 = \sqrt{\lambda_1} \leq \|A\|_F \leq n\mu. \tag{15}$$

An obvious estimation for $\mathrm{Cond}_2(A)$ uses

$$\mathrm{Cond}_2(A) = \frac{\sigma_1}{\sigma_n} = \frac{\sigma_1 \prod_{i=1}^{n-1} \sigma_i}{\prod_{i=1}^{n} \sigma_i} = \frac{\sigma_1 \prod_{i=1}^{n-1} \sigma_i}{|\det(A)|} = \sigma_1 \prod_{i=1}^{n-1} \sigma_i \leq \sigma_1^n \leq n^n \mu^n. \tag{16}$$

This can be improved as follows, in particular allowing to use information not only on the maximum size of the elements $|a_{ij}|$, but also on their individual size and distribution. For any positive definite $n \times n$-matrix B, Hadamard's determinant inequality [Horn/Johnson: Matrix Analysis, Theorem 7.8.1] yields

$$\det(B) \leq \prod_{i=1}^{n} b_{ii}.$$

For any matrix B with $|\det(B)| = 1$ we have $|(B^{-1})_{ii}| = |\det(B(i))|$, where $B(i)$ is the matrix B after deleting the i-th row and column. Applying this to the positive definite matrix $B := AA^T$ and using that $B(i)$ is, as a principal submatrix, positive definite as well, shows

$$((AA^T)^{-1})_{ii} \leq \prod_{j \neq i} (AA^T)_{jj} \leq (n\mu^2)^{n-1}, \tag{17}$$

so that

$$\|A^{-1}\|_2^2 = \|(AA^T)^{-1}\|_2 \leq \mathrm{tr}((AA^T)^{-1}) \leq n^n \mu^{2(n-1)}. \tag{18}$$

Putting things together yields the estimation

$$\mathrm{Cond}_2(A) = \|A\|_2 \|A^{-1}\|_2 \leq n^{n/2+1} \mu^n. \tag{19}$$

An even better estimation is obtained by

Theorem 1**(Guggenheimer, et al. [14], 1995)**: For an arbitrary real $n \times n$ matrix A we have

$$\mathrm{Cond}_2(A) < \frac{2}{|\det(A)|} \left(\frac{\|A\|_F}{n} \right)^{\frac{n}{2}}. \tag{20}$$

Using $|\det(A)| = 1$ and $\|A\|_F \leq n^2 \mu^2$ yields for our case

$$\mathrm{cond}_2(A) < 2n^{n/2} \mu^n. \tag{21}$$

This estimation seems almost sharp as for matrices with a singular value distribution of AA^T like

$$\lambda_1 \approx 2\gamma \quad \text{and} \quad \lambda_2 = \lambda_3 = \ldots = \lambda_{n-1} \approx \gamma \tag{22}$$

for some constant γ, and choosing λ_n according to $\det(AA^T) = \prod \lambda_i = 1$. Moreover, from Eqs. (2) and (13) we have for $n \geq 2$ that

$$\lambda_1 \lambda_2 \lambda_3 \cdots \lambda_n = 1 \tag{23}$$

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n = \mathrm{tr}(AA^T) \tag{24}$$

$$\mathrm{Cond}_2^2(A) = \frac{\lambda_1}{\lambda_n}, \quad \text{i.e.,} \quad \mathrm{Cond}_2(A) = \sqrt{\frac{\lambda_1}{\lambda_n}} = \frac{\sigma_1}{\sigma_n}. \tag{25}$$

Note that all estimations so far are valid for any *real* matrix satisfying the assumptions (9). Bounds for the condition number of *integer* matrices satisfying (9) may be sharper, however, we think that the difference is not too big.

A very large condition number as in (21) is achieved for a specific singular value distribution as in (22). So we may ask:

Question 1: If the distribution of $\lambda_i (i = 1, 2, \cdots, n)$ differs significantly from Eq. (22), how does the upper bound of $\mathrm{Cond}_2(A)$ change?

Next we try to examine the relation between the distribution of λ_i and an upper bounds of $\mathrm{Cond}_2(A)$ subject to $|\det(A)| = 1$ and $|a_{ij}| \leq \mu$. We consider the following special distributions of eigenvalues λ_i.

1. **Case where $\lambda_{1 \ldots n}$ take only two distinct values, i.e.,**

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k, \quad \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n. \tag{26}$$

In particular we consider some special case of Eq. (26):

1a. Case of $k = 1$, i.e.,

$$\lambda_1 > \lambda_2 = \lambda_3 = \cdots = \lambda_n, \tag{27}$$

1b. Case of $k = n/2$ (provided n is even)

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n/2}, \quad \lambda_{n/2+1} = \lambda_{n/2+2} = \cdots = \lambda_n, \tag{28}$$

1c. Case of $k = n - 1$, i.e.,

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} > \lambda_n. \tag{29}$$

2. **Case where $\lambda_{1 \ldots n}$ have logarithmically uniform distribution, i.e.,**

$$\lambda_1 = r^2 \lambda_2 = r^4 \lambda_3 = \cdots = r^{2i} \lambda_{i+1} = \cdots = r^{2(n-1)} \lambda_n \quad (r > 1). \tag{30}$$

3. **Case where $\lambda_{1 \ldots n}$ take only three distinct values as**

$$\lambda_1 = \lambda_2 = \cdots = \lambda_k > \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_{n-l} = 1 > \lambda_{n-l+1} = \lambda_{n-l+2} = \cdots = \lambda_n, \tag{31}$$

Problem 1: For each eigenvalue distribution in Eqs. (26)–(31) determine $\mathrm{Cond}_2(A)$ in Eq. (25) under the restrictions (9).

In these special types of eigenvalue distributions (Eqs. (26)–(31)) there are two independent variables, i.e., λ_1 and λ_n for example, in Cases (1) and (3) and r and λ_1 in Case (2). On the other hand there are two equations, i.e., Eqs. (23) and (24). So these equations can be solved for λ_i in principle and the condition number $\mathrm{Cond}_2(A)$ in Eqs. (25) can also be determined.

For example in Case 1 (with restriction (26)) we can proceed as follows. We have

$$1 = |\det(A)| = \prod_{i=1}^{n} \lambda_i = \lambda_1^k \lambda_n^{n-k}, \tag{32}$$

so that

$$\mathrm{Cond}_2(A)^2 = \frac{\lambda_1}{\lambda_n} = \lambda_1^{1+\frac{k}{n-k}} = \lambda_1^{\frac{n}{n-k}}. \tag{33}$$

Using (24) and (14) yields

$$k\lambda_1 \le n^2\mu^2 - (n-k)\lambda_n \le n^2\mu^2 \tag{34}$$

and proves

$$\mathrm{Cond}_2(A) \le [\frac{n\mu}{\sqrt{k}}]^{\frac{n}{n-k}}. \tag{35}$$

This implies that with increasing value of k the maximally achievable condition number decreases substantially. For example, for $k \approx n/2$, i.e. about $n/2$ large singular values, matrices satisfying (9) can have condition numbers only up to about $2n\mu^2$.

The rigorous analytical solution for λ_i as well as $\mathrm{Cond}_2(A)$ seems difficult. However, we are only interested in a reasonable estimate, so we give only upper bounds of $\mathrm{Cond}_2(A)$ in Table I. The quantities in Column 2 of Table I show analytical upper bounds of $\mathrm{Cond}_2(A)$ for each case. Note that they are almost the same as the rigorous solutions of Problem 1, in particular when n is large.

Table I. Upper bounds for typical λ_i distributions.

Cases	Upper bounds in terms of μ	Upper bounds for sparse cases
Optimum case (Eq. (22))	$2n^{n/2}\mu^n$	$2c^{n/2}\mu^n$
Case 1 (Eq. (25))	$\left(\frac{n\mu}{\sqrt{k}}\right)^{\frac{n}{n-k}}$	$\left(\frac{cn\mu^2}{k}\right)^{\frac{n}{2(n-k)}}$
Case 1a (Eq. (27))	$n\mu$ $(n \gg 1)$	$\sqrt{cn}\mu$ $(n \gg 1)$
Case 1b (Eq. (28))	$2n\mu^2$	$2c\mu^2$
Case 1c (Eq. (29))	$n^{n/2}\mu^n$ $(n \gg 1)$	$c^{n/2}\mu^n$ $(n \gg 1)$
Case 2 (Eq. (30))	$n^2\mu^2$	$cn\mu^2$
Case 3 (Eq. (31))	$\left(\frac{n\mu}{\sqrt{k}}\right)^{\frac{k+l}{l}}$	$\left(\frac{cn\mu^2}{k}\right)^{\frac{k+l}{2l}}$
$k = l$ in Case 3	$\frac{n^2\mu^2}{k}$	$\frac{cn\mu^2}{k}$

In some entries of Table I a comment "$n \gg 1$" occurs. It means that the displayed bound is essentially true up to a nasty factor like $\sqrt[n]{n}$ which is near 1 for larger values of n.

In the above table the Case 2 is of specific interest. This reflects a geometric distribution of the singular values, from the largest to the smallest. In some sense this seems particularly desirable, it is also the default in Matlab's "randsvd" function to generate random-like matrices with a specified condition number. In this case (30) implies

$$\mathrm{Cond}_2(A)^2 = \frac{\lambda_1}{\lambda_n} = r^{2(n-1)}. \tag{36}$$

To estimate r we set $\alpha := r^{-2}$ and have

$$1 = |\det(A)| = \prod_{i=1}^{n} \lambda_i = \lambda_1 \cdot \alpha\lambda_1 \cdot \ldots \cdot \alpha^{n-1}\lambda_1 = \lambda_1^n \alpha^{\frac{n(n-1)}{2}}, \tag{37}$$

so that $\alpha = r^{-2}$ and (14) imply

$$r^{n(n-1)} = \lambda_1^n \le (n^2\mu^2)^n \tag{38}$$

and therefore

$$\mathrm{Cond}_2(A) \le n^2\mu^2. \tag{39}$$

In other words, for a matrix with integer entries limited by 10^{16} the maximally achievable condition number is about $10^{32}n^2$, so there are no such extremely ill-conditioned matrices with this desirable distribution of singular values. The estimation for Case 3 is similarly derived.

3.2 Upper bounds of condition number of sparse matrices

For sparse matrices as mentioned in footnote 2 on page 2, i.e. few entries ± 1, about c entries of size μ and otherwise zero entries, $\beta = \|A\|_F^2$ is limited by about $cn\mu^2$ rather than $n^2\mu^2$. Inserting this into (20) improves the estimation of the maximal condition number into

$$\mathrm{Cond}_2(A) < 2c^{n/2}\mu^n \quad \text{where } |\det(A)| = 1 \tag{40}$$

as shown in the third column of the first row of Table I. Note that the right-most quantity in Eq. (40) is extremely large for large n.

Estimations for the largest admissible condition number for sparse matrix pattern and depending on the distribution of the singular values are shown in the rightmost column of Table I. Practical experience suggests that these bounds are almost sharp.

From Table I we see that

1. If the distribution of singular values of the matrices A and A^{-1} are somehow similar (i.e, in Cases 1b, Case 2 and Case 3($k = l$)), then the maximally achievable condition number is rather small (only about μ^2).
2. Extremely large condition numbers may only be realized when the number of small eigenvalues is considerably smaller compared to the number of large eigenvalues.
3. In particular the desirable geometric distribution of singular values does not allow extremely large condition numbers unless μ, i.e. the maximum absolute value of the matrix entries, is extremely large.

4. Generation of ill-conditioned matrix similar to companion matrix

4.1 Generation method

In this section we propose a generation method of an ill-conditioned matrix which has a similar form to a companion matrix and which can be regarded as a half size of that in Rump [7].

Let A be an $n \times n$ *integer* matrix:

$$A = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{n-1} & a_n \\ 1 & -\nu_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\nu_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & -\nu_3 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -\nu_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\nu_{n-1} \end{bmatrix} \tag{41}$$

Without loss of generality we can assume[7]

$$\nu_i > 0 \quad (i = 1, 2, \cdots, n-1) \tag{42}$$

We also assume

$$0 < \nu_i \le \mu \ \ (i = 1, 2, \cdots, n-1), \quad |a_i| \le \mu \ \ (i = 1, 2, \cdots, n) \tag{43}$$

We determine a_i $(i = 1, \cdots, n)$ such that

$$((((a_1\nu_1 + a_2)\nu_2 + a_3)\nu_3 + a_4)\nu_4 + \cdots)\nu_{n-1} + a_n = 1 \tag{44}$$

This condition corresponds to the Pell equation in [7]. Referring to Eq. (44), we will describe how to determine a_i.

Step 1: From Eq. (44) we see that $1 - a_n$ must be divided by ν_{n-1}, i.e.,

$$1 - a_n \equiv 0 \pmod{\nu_{n-1}} \tag{45}$$

[7] If $\nu_1 < 0$, then we can change it into positive by multiplying both the second column and the third row by -1. By applying a similar operation to ν_2, ν_3, $\cdots$, ν_{n-1} **in this order**, we can make all ν_i positive.

We define k_{n-1} as

$$\frac{1-a_n}{\nu_{n-1}} \equiv k_{n-1} \tag{46}$$

where k_{n-1} may be 0, ±1, ±2, ±3, $\cdots$ and by Eqs. (46) and (43)

$$\begin{aligned} a_n &= 1-\nu_{n-1}k_{n-1} & (47)\\ |a_n| &= |1-\nu_{n-1}k_{n-1}| \leq \mu & (48) \end{aligned}$$

have to be satisfied. From Eq. (48) we have

$$-\mu \leq 1-\nu_{n-1}k_{n-1} \leq \mu \qquad \therefore\ -\mu+1 \leq \nu_{n-1}k_{n-1} \leq 1+\mu \tag{49}$$

from which we have

$$\frac{-\mu+1}{\nu_{n-1}} \leq k_{n-1} \leq \frac{1+\mu}{\nu_{n-1}} \tag{50}$$

Though we can choose k_{n-1} satisfying Eq. (50) arbitrarily,

$$k_{n-1} = \left[\frac{1+\mu}{2\nu_{n-1}}\right] (>0) \ \text{ and } \ \left[\frac{1-\mu}{2\nu_{n-1}}\right] (<0) \quad ([\cdot] \text{ means the Gauss notation}) \tag{51}$$

are reasonable candidates for k_{n-1}. Then a_n is determined by Eq. (47).

Step 2: Quite similarly we can derive equations corresponding to Eqs. (46)–(51).

For convenience let

$$k_n = 1 \tag{52}$$

Then we can calculate a_j and k_j recursively for $j = n-1, n-2, \cdots, 2$ in the descent order as follows:

$$\begin{aligned} & k_{j+1} - a_{j+1} \equiv 0 \pmod{\nu_j} & (53)\\ & \frac{k_{j+1}-a_{j+1}}{\nu_j} \equiv k_j, \quad (k_j = 0, \pm1, \pm2, \cdots) & (54)\\ & a_{j+1} = k_{j+1} - \nu_j k_j & (55)\\ & |a_{j+1}| = |k_{j+1} - \nu_j k_j| < \mu & (56)\\ & \frac{-\mu + k_{j+1}}{\nu_j} < k_j < \frac{k_{j+1}+\mu}{\nu_j} & (57) \end{aligned}$$

Similarly

$$k_j = \left[\frac{k_{j+1}+\mu}{2\nu_j}\right] (>0) \ \text{ and } \ \left[\frac{k_{j+1}-\mu}{2\nu_j}\right] (<0) \tag{58}$$

are appropriate candidate of k_i. Equation (58) is one of choices but we can choose k_j arbitrarily in Eq. (57).

Note that the case of $j = n-1$ in Eqs. (53)–(58) corresponds to Step 1.

Step 3: Finally let

$$a_1 \equiv k_1 \tag{59}$$

Since $\nu_i > 0$ holds, some (but not all) of a_i are necessarily negative (See Eq. (44)). We choose the sign of a_i, for example, as

$$a_{2i} > 0,\ a_{2i+1} < 0, \quad \therefore\ k_{2i} < 0,\ k_{2i+1} > 0 \quad (i = 1, 2, \cdots,) \tag{60}$$

or

$$a_{2i} < 0,\ a_{2i+1} > 0, \quad \therefore\ k_{2i} > 0,\ k_{2i+1} < 0 \quad (i = 1, 2, \cdots,) \tag{61}$$

4.2 Condition number of A in Eq. (41)

In this section we evaluate the condition number of A in Eq. (41) with the infinity-norm. For this purpose we first calculate the inverse matrix A^{-1}. It can be calculated in a similar way as in [8] as follows:

Let

$$H \equiv \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & \prod_1^{n-1} \nu_i \\ 0 & 1 & 0 & \cdots & 0 & 0 & \prod_2^{n-1} \nu_i \\ 0 & 0 & 1 & \cdots & 0 & 0 & \prod_3^{n-1} \nu_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \nu_{n-2}\nu_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & \nu_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix} \tag{62}$$

Then we have

$$A' \equiv AH \tag{63}$$

$$= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & \cdots & a_{n-2} & a_{n-1} & a_n \\ 1 & -\nu_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\nu_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -\nu_3 & 0 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -\nu_{n-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -\nu_{n-1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & \prod_1^{n-1} \nu_i \\ 0 & 1 & 0 & 0 & \cdots & 0 & \prod_2^{n-1} \nu_i \\ 0 & 0 & 1 & 0 & \cdots & 0 & \prod_3^{n-1} \nu_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \nu_{n-2}\nu_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & \nu_{n-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & \cdots & a_{n-2} & a_{n-1} & 1 \\ 1 & -\nu_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -\nu_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -\nu_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -\nu_{n-2} & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \tag{64}$$

Let

$$A' = \begin{bmatrix} U & 1 \\ W & 0 \end{bmatrix} \tag{65}$$

$$U = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-1} \end{bmatrix} \tag{66}$$

$$W = \begin{bmatrix} 1 & -\nu_1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -\nu_2 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -\nu_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & -\nu_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{67}$$

Then

$$(A')^{-1} = \begin{bmatrix} 0 & W^{-1} \\ 1 & -UW^{-1} \end{bmatrix} \tag{68}$$

Here

$$W^{-1} = \begin{bmatrix} 1 & \nu_1 & \nu_1\nu_2 & \prod_1^3 \nu_i & \cdots & \prod_1^{n-2} \nu_i \\ 0 & 1 & \nu_2 & \nu_2\nu_3 & \cdots & \prod_2^{n-2} \nu_i \\ 0 & 0 & 1 & \nu_3 & \cdots & \prod_3^{n-2} \nu_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \nu_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{69}$$

$$
\begin{aligned}
-UW^{-1} &= -\left[\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_{n-1} \end{array}\right] \left[\begin{array}{cccccc} 1 & \nu_1 & \nu_1\nu_2 & \prod_1^3 \nu_i & \cdots & \prod_1^{n-2} \nu_i \\ 0 & 1 & \nu_2 & \nu_2\nu_3 & \cdots & \prod_2^{n-2} \nu_i \\ 0 & 0 & 1 & \nu_3 & \cdots & \prod_3^{n-2} \nu_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & \nu_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{array}\right] \\
&\equiv -\left[\begin{array}{ccccc} K_1, & K_2, & K_3, & \cdots, & K_{n-1} \end{array}\right] \qquad (70)
\end{aligned}
$$

where

$$
K_j \equiv a_1 \prod_{i=1}^{j-1} \nu_i + a_2 \prod_{i=2}^{j-1} \nu_i + \cdots + a_j \quad (j = 1, 2, \cdots, n-1) \qquad (71)
$$

i.e.,

$$
\left.\begin{array}{rcl} K_1 &=& a_1 \\ K_2 &=& a_1\nu_1 + a_2 \\ K_3 &=& a_1\nu_1\nu_2 + a_2\nu_2 + a_3 \\ & \vdots & \\ K_{n-1} &=& a_1 \prod_{i=1}^{n-2} \nu_i + \cdots + a_{n-1} \end{array}\right\} \qquad (72)
$$

So we have the final form of A^{-1} as:

$$
\begin{aligned}
A^{-1} &= H(A')^{-1} \qquad (73) \\
&= \left[\begin{array}{ccccccc} 1 & 0 & 0 & \cdots & 0 & 0 & \prod_1^{n-1} \nu_i \\ 0 & 1 & 0 & \cdots & 0 & 0 & \prod_2^{n-1} \nu_i \\ 0 & 0 & 1 & \cdots & 0 & 0 & \prod_3^{n-1} \nu_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 & 0 & \nu_{n-2}\nu_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 1 & \nu_{n-1} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right] \\
&\times \left[\begin{array}{ccccccc} 0 & 1 & \nu_1 & \nu_1\nu_2 & \cdots & \prod_1^{n-3} \nu_i & \prod_1^{n-2} \nu_i \\ 0 & 0 & 1 & \nu_2 & \cdots & \prod_2^{n-3} & \prod_2^{n-2} \nu_i \\ 0 & 0 & 0 & 1 & \cdots & \prod_3^{n-3} & \prod_3^{n-2} \nu_i \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & \nu_{n-2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & -K_1 & -K_2 & -K_3 & \cdots & -K_{n-2} & -K_{n-1} \end{array}\right] \\
&= \left[\begin{array}{ccccc} \prod_1^{n-1} \nu_i & 1 - K_1 \prod_1^{n-1} \nu_i & \nu_1 - K_2 \prod_1^{n-1} \nu_i & \cdots & \prod_1^{n-2} \nu_i - K_{n-1} \prod_1^{n-1} \nu_i \\ \prod_2^{n-1} \nu_i & -K_1 \prod_2^{n-1} \nu_i & 1 - K_2 \prod_2^{n-1} \nu_i & \cdots & \prod_2^{n-2} \nu_i - K_{n-1} \prod_2^{n-1} \nu_i \\ \prod_3^{n-1} \nu_i & -K_1 \prod_3^{n-1} \nu_i & -K_2 \prod_3^{n-1} \nu_i & \cdots & \prod_3^{n-2} \nu_i - K_{n-1} \prod_3^{n-1} \nu_i \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \nu_{n-1} & -K_1\nu_{n-1} & -K_2\nu_{n-1} & \cdots & 1 - K_{n-1}\nu_{n-1} \\ 1 & -K_1 & -K_2 & \cdots & -K_{n-1} \end{array}\right] \qquad (74)
\end{aligned}
$$

By the way we can show that

$$
k_i = K_i \quad (i = 1, 2, \cdots, n-1) \qquad (75)
$$

which can be easily derived by comparing Eq. (74) with Eq. (54).

Since we can verify that the magnitude of the $(1, j)$ element of the matrix in Eq. (74) is larger than that of (i, j) $(i \geq 2)$ element, we have

$$
\begin{aligned}
\|A^{-1}\|_\infty &= \prod_1^{n-1}\nu_i + \left|1-K_1\prod_1^{n-1}\nu_i\right| + \left|\nu_1 - K_2\prod_1^{n-1}\nu_i\right| + \cdots + \left|\prod_1^{n-2}\nu_i - K_{n-1}\prod_1^{n-1}\nu_i\right| \\
&= (|k_1|+|k_2|+\cdots+|k_{n-1}|)\prod_1^{n-1}\nu_i - \sum_1^{n-1}\frac{k_i}{|k_i|}\nu_i \qquad (76) \\
\|A\|_\infty &= \max\left\{\sum_1^n |a_i|, \quad \max_i(\nu_i+1)\right\} \qquad (77)
\end{aligned}
$$

Equation (76) can be obtained from the first row of Eq. (74).
Finally we have

$$
\begin{aligned}
\mathrm{Cond}_\infty(A) &= \max\left\{\sum_1^n |a_i|, \quad \max_i(\nu_i+1)\right\}\cdot\left\{\sum_1^{n-1}|k_i|\prod_1^{n-1}\nu_i - \sum_1^{n-1}\frac{k_i}{|k_i|}\nu_i\right\} \qquad (78) \\
&\geq \sum_1^n |a_i|\left(\sum_1^{n-1}|k_i|\prod_1^{n-1}\nu_i - \sum_1^{n-1}\nu_i\right) \qquad (79)
\end{aligned}
$$

This corresponds to the Rump's result in Eq. (5). If we choose $k_i \neq 0$, then $\sum_1^{n-1}|k_i| \geq n-1$. In addition if $\nu_i \gg 1$, then

$$
\text{The right-hand side of Eq. (79)} \approx \sum_1^n |a_i|\sum_1^{n-1}|k_i|\prod_1^{n-1}\nu_i \geq (n-1)\sum_1^n|a_i|\prod_1^{n-1}\nu_i \qquad (80)
$$

From Eqs. (76)–(80) we have a *very rough estimation* on $\mathrm{Cond}_2(A)$ as

$$
\mathrm{Cond}_\infty(A) \quad \begin{cases} > (n-1)^2\mu^{n-1} & \text{if } \nu_i \sim \mu, ||a_i| \sim \mu \\ > (n-1)\mu^{\frac{n+1}{2}} & \text{if } |a_i| \sim \mu, \quad \nu_i \sim \sqrt{\mu} \end{cases}
$$

This means that Eq. (81) are approximately same but are a little inferior to that in [7].

4.3 Considerations through examples

Example 1: Let

$$
n=4, \quad \mu=10, \quad \nu_1=\nu_2=\nu_3=5 \qquad (81)
$$

We will choose k_i and a_i according to Eqs. (53)–(58). Since $1-a_4$ must be divided by $(\nu_3=)5$, we have $a_4 = 1, -4, 6, -9, \cdots$. So we choose $a_4=-9$, for example. Then we have $k_3=(1-a_4)/5=2$.

Since k_3-a_3 must be divided by $(\nu_2=)5$, we choose $a_3=7$ and then $k_2=-1$. Since (k_2-a_2) must be divided by $(\nu_1=)5$, we choose as $a_2=-6$ and then $k_1=a_1=1$. Indeed we have $((1\times 5+(-6))\times 5+7)\times 5+(-9))=1$.

We therefore have

$$
A = \begin{bmatrix} 1 & -6 & 7 & -9 \\ 1 & -5 & 0 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 1 & -5 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 125 & -124 & 130 & -225 \\ 25 & -25 & 26 & -45 \\ 5 & -5 & 5 & -9 \\ 1 & -1 & 1 & -2 \end{bmatrix} \qquad (82)
$$

from which we see that

$$
\mathrm{Cond}_\infty(A) = \|A\|_\infty\cdot\|A^{-1}\|_\infty = 13892 \qquad (83)
$$

This nearly agrees with Eq. (81). Singular values of A are 14.109, 5.136, 4.421 and 0.003121. Then we have

$$
\mathrm{Cond}_2(A) \approx \frac{14.109}{0.003121} \approx 4520.3
$$

Thus A has a considerably large condition number even for such small μ and n.

Example 2: Let

$$n = 4, \quad \mu = 1000, \quad \nu_1 = \nu_2 = \nu_3 = 50 \tag{84}$$

In a quite similar way as in Example 1 we will choose k_i and a_i according to Eqs. (53)–(58). We have $a_4 = -799$, $a_3 = 716$, $a_2 = -864$, and $a_1 = 17$ as one of choices, i.e.,

$$A = \begin{bmatrix} 17 & -864 & 716 & -799 \\ 1 & -50 & 0 & 0 \\ 0 & 1 & -50 & 0 \\ 0 & 0 & 1 & -50 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 125000 & -2124999 & 1750050 & -1997500 \\ 2500 & -42500 & 35001 & -39950 \\ 50 & -850 & 700 & -799 \\ 1 & -17 & 14 & -16 \end{bmatrix} \tag{85}$$

from which we see that

$$\mathrm{Cond}_\infty(A) = \|A\|_\infty \cdot \|A^{-1}\|_\infty \approx 13 \times 10^8 \tag{86}$$

Singular values are 1378.6, 50.01, 49.38 and 0.0000003. Then we have

$$\mathrm{Cond}_2(A) = \frac{1378.6}{0.0000003} \approx 4.693 \cdot 10^9$$

Thus the condition number of A is very large.

Note that in the above example A has three large singular values and one very small one singular value, and A^{-1} is near to a matrix with rank one. As another example, we consider the case where $\nu_i = 1$ $(i = 1, \cdots, n-1)$. Example 3 is the result for $n = 4$, $\mu = 1000$ and $\ni_i = 1 (i = 1, \cdots, n)$.

Example 3:

$$A = \begin{bmatrix} 300 & -590 & 850 & -561 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} -1 & 301 & -289 & 561 \\ -1 & 300 & -289 & 561 \\ -1 & 300 & -290 & 561 \\ -1 & 300 & -290 & 560 \end{bmatrix} \tag{87}$$

$$\mathrm{Cond}_\infty(A) \approx 1.6977 \times 10^6$$

Singular values are 1214.6, 1.438, 0.8003, and 0.0007155.

As is expected, $\lambda_1 \gg \lambda_2 \approx \lambda_3 \gg \lambda_4$ holds but even in this case A^{-1} is near to a matrix with rank one, since the smallest eigenvalue is extremely small compared to other ones.

5. Generation of more desirable ill-conditioned matrices

5.1 Preliminary

As seen from the above examples, inverse matrices with extremely large condition number are very close to a matrix with rank one. One of the authors showed a more interesting example in which all elements of A^{-1} have almost identical value. This property seems undesirable as a benchmark matrix for verified numerical algorithm. In this section we propose some desirable properties for singular values of A and give partial solutions to it.

Let an $n \times n$ integer matrix be A, where $n = 2m$. We would like to generate A such that the condition number of A is very large and simultaneously that singular values of A have a desirable distribution. Let singular values of A be σ_i $(= \sqrt{\lambda_i})$ $(i = 1, 2, \cdots, n)$ where σ_i and λ_i satisfy Eq. (12).

5.2 Desirable properties of *A*

The definition of the condition number $\|A\|\|A^{-1}\|$ may suggest that it seems desirable that the singular values of both A and A^{-1} have the same property. Therefore we consider that Cases (i)–(iii) below are desirable as ill-conditioned benchmark matrices.

$$\text{Case (i)} \qquad \sigma_1 = \sigma_2 = \sigma_3 = \cdots = \sigma_m > 1 > \sigma_{m+1} = \sigma_{m+2} = \cdots = \sigma_{2m} \tag{88}$$

$$\text{Case (ii)} \qquad \sigma_1 = r\sigma_2 = r^2\sigma_3 = \cdots = r^{2m-2}\sigma_{2m-1} = r^{2m-1}\sigma_{2m} \quad (r > 1) \tag{89}$$

$$\text{Case (iii)} \qquad \sigma_1 = \cdots = \sigma_l > \sigma_{l+1} = \cdots = \sigma_{n-l} = 1 > \sigma_{n-l+1} = \cdots = \sigma_n \ll 1 \tag{90}$$

We call Cases (i), (ii) and (iii) "*singular values with two levels* (simply *2L case*)", "*logarithmically uniform singular value case* (simply *logarithmic case*)", and "*singular values with three levels* (simply

3L case)", respectively. Note that if A is an integer matrix, then Eq. (89) may be satisfied only approximately.

Then our problem is as follows:

Problem 2: Generate a variety of matrices satisfying Eqs. (88), (89) or (90) and making their condition number as large as possible.

Readers may know that there is a standard tool ("*randsvd*") in MATLAB to generate Eq. (89). However due to the rounding a large condition number greater than 10^{16} are usually not generated by "randsvd". In this paper we aim [8] to generate integer matrices with about $10^{16} < \text{Cond}(A) < 10^{32}$.

The subsequent results are partial solutions for Problem 2.

5.3 Matrix in consideration

To realize Eqs. (88), (89) and (90) we heuristically utilize the following special type of $n \times n$ matrices A where $n = 2m$.

$$A = \begin{bmatrix} \mathbf{1} & B \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{91}$$

Here $\mathbf{1}$ is a unit matrix of order m and $B = [b_{ij}]$ with $|b_{ij}| \le \mu$ is an $m \times m$ matrix[9]. Apparently A satisfies $\det(A) = 1$.

Then we have

$$A^{-1} = \begin{bmatrix} \mathbf{1} & -B \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \tag{92}$$

We have:

$$\|A\|_\infty = \|A^{-1}\|_\infty = 1 + \|B\|_\infty \le m\mu + 1 \quad (|b_{ij}| \le \mu) \tag{93}$$

and therefore

$$\text{Cond}_\infty(A) = \|A\|_\infty \|A^{-1}\|_\infty = \|A\|_\infty^2 \le (m\mu + 1)^2 = \left(\frac{n}{2}\mu + 1\right)^2 \tag{94}$$

Comparing this with Table I in Section 3, we guess that A in Eq. (91) may possibly realize near the maximum condition number. Since

$$AA^T = \begin{bmatrix} \mathbf{1} + BB^T & B \\ B^T & \mathbf{1} \end{bmatrix}, \quad A^{-1}(A^{-1})^T = \begin{bmatrix} \mathbf{1} + BB^T & -B \\ -B^T & \mathbf{1} \end{bmatrix}, \tag{95}$$

AA^T and $A^{-1}(A^{-1})^T$ have the same eigenvalues. Therefore we have:

Lemma 1: For the matrix A in Eq. (91) we have

$$\lambda_i = \frac{1}{\lambda_{n-i}} \quad \text{i.e.,} \quad \sigma_i = \frac{1}{\sigma_{n-i}} \quad (i = 1, 2, \cdots, m) \tag{96}$$

Example 4:

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 100 & 300 & -600 & 200 \\ 0 & 1 & 0 & 0 & 500 & -400 & 300 & -200 \\ 0 & 0 & 1 & 0 & 100 & 300 & -600 & 200 \\ 0 & 0 & 0 & 1 & -800 & 900 & -100 & -700 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{97}$$

$$\text{Cond}_\infty(A) = 2274745.2$$

Singular values are $\sigma_1 = 1508.2$, $\sigma_2 = 1030.0$, $\sigma_3 = 392.81$, $\sigma_4 = 1$, $\sigma_5 = 1 = 1/\sigma_4$, $\sigma_6 = 0.0025458 = 1/\sigma_3$, $\sigma_7 = 0.0009708 = 1/\sigma_2$, and $\sigma_8 = 0.0006630 = 1/\sigma_1$. Note that in this example $\sigma_4 = \sigma_5 = 1$.

Concerning the singular values with $\sigma_i = 1$, see Lemma 2 in the subsequent subsection.

[8] Note that we cannot generate a matrix with extremely large condition number, i.e., $\text{Cond}(A) > 10^{40}$ because of the limitation due to the third and the fourth columns in Table I.

[9] We can derive similar results as Eqs. (92)–(115) below when B in Eq. (91) is **a rectangular matrix** but we will omit it. For our purpose B should be an integer matrix but in the most part of this section B is assumed as a real matrix.

5.4 Singular values of B and A

Let the singular value decomposition of B be

$$B = U\Sigma_B V^T \tag{98}$$

where U and V are orthogonal matrices and

$$\Sigma_B = \text{diag}\,[\sigma_{B1}, \sigma_{B2}, \sigma_{B3}, \cdots, \sigma_{Bm}] \tag{99}$$
$$\sigma_{B1} \geq \sigma_{B2} \geq \sigma_{B3} \geq \cdots \geq \sigma_{Bm} \geq 0 \tag{100}$$

We therefore have

$$\Sigma_B = U^{-1}B(V^T)^{-1} \tag{101}$$

and

$$A = \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \mathbf{1} & \Sigma_B \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} U^T & 0 \\ 0 & V^T \end{bmatrix} \tag{102}$$

$$\begin{aligned} AA^T &= \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \mathbf{1} & \Sigma_B \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \begin{bmatrix} U^T & 0 \\ 0 & V^T \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \Sigma_B & \mathbf{1} \end{bmatrix} \begin{bmatrix} U^T & 0 \\ 0 & V^T \end{bmatrix} \\ &= \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} \mathbf{1}+\Sigma_B^2 & \Sigma_B \\ \Sigma_B & \mathbf{1} \end{bmatrix} \begin{bmatrix} U^T & 0 \\ 0 & V^T \end{bmatrix} \end{aligned} \tag{103}$$

Let

$$K \equiv \begin{bmatrix} \mathbf{1}+\Sigma_B^2 & \Sigma_B \\ \Sigma_B & \mathbf{1} \end{bmatrix} \tag{104}$$

$$= \left[\begin{array}{ccc|ccc} 1+\sigma_{B1}^2 & \cdots & 0 & \sigma_{B1} & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & 1+\sigma_{Bm}^2 & 0 & \cdots & \sigma_{Bm} \\ \hline \sigma_{B1} & \cdots & 0 & 1 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & \sigma_{Bm} & 0 & \cdots & 1 \end{array}\right] \tag{105}$$

We will calculate the eigenvalues of K, i.e., those of AA^T. Let the characteristic polynomial of K be $\phi_K(\lambda)$. Thus

$$\begin{aligned} \phi_K(\lambda) &= \det(\lambda\mathbf{1} - K) \\ &= \det\left(\left[\begin{array}{ccc|ccc} \lambda-(1+\sigma_{B1}^2) & \cdots & 0 & -\sigma_{B1} & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & \lambda-(1+\sigma_{Bm}^2) & 0 & \cdots & -\sigma_{Bm} \\ \hline -\sigma_{B1} & \cdots & 0 & \lambda-1 & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & -\sigma_{Bm} & 0 & \cdots & \lambda-1 \end{array}\right]\right) & (106) \\ &= \det\left(\begin{bmatrix} \lambda-(1+\sigma_{B1}^2) & -\sigma_{B1} \\ -\sigma_{B1} & \lambda-1 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} \lambda-(1+\sigma_{Bm}^2) & -\sigma_{Bm} \\ -\sigma_{Bm} & \lambda-1 \end{bmatrix}\right) & (107) \\ &= \prod_{i=1}^{m} \left[\lambda^2 - (2+\sigma_{Bi}^2)\lambda + 1\right] & (108) \end{aligned}$$

Here $\oplus$ means the direct sum of matrices.

Therefore the eigenvalues of K are given as solutions of

$$\lambda^2 - (2+\sigma_{Bi}^2)\lambda + 1 = 0 \quad (i = 1, 2, \cdots, m) \tag{109}$$

Let the solutions of Eq. (109) be

$$\lambda_{i\pm} = \frac{2+\sigma_{Bi}^2 \pm \sqrt{(2+\sigma_{Bi}^2)^2-4}}{2} = \frac{2+\sigma_{Bi}^2 \pm \sigma_{Bi}\sqrt{\sigma_{Bi}^2+4}}{2} \quad (i=1,2,\cdots,m) \tag{110}$$

i.e.,

$$\lambda_{i+} = \frac{2+\sigma_{Bi}^2 + \sigma_{Bi}\sqrt{\sigma_{Bi}^2+4}}{2}, \quad \lambda_{i-} = \frac{2+\sigma_{Bi}^2 - \sigma_{Bi}\sqrt{\sigma_{Bi}^2+4}}{2} \tag{111}$$

Note that

$$\lambda_{1+} \geq \lambda_{2+} \geq \cdots \geq \lambda_{m+} \geq 1 \geq \lambda_{m-} \geq \lambda_{(m-1)-} \geq \cdots \geq \lambda_{1-} (>0) \tag{112}$$

$$\lambda_{i+}\lambda_{i-} = 1 \tag{113}$$

The singular values σ_i ($\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{2m}$) of A are given as

$$\sigma_i = \sqrt{\lambda_{i+}} \quad (i=1,2,\cdots,m) \tag{114}$$

$$\sigma_i = \sqrt{\lambda_{(2m-i+1)-}} \quad (i=m+1,m+2,\cdots,2m) \tag{115}$$

Eqs. (111), (114) and (115) show the relation between the singular values ($=\sigma_i(i=1,\cdots,n)$) of A and those ($=\sigma_{Bj}$)($j=1,\cdots,n$) of B.

By the way we see from Eq. (111) that $\lambda_{i+}=\lambda_{i-}=1$ if and only if $\sigma_{Bi}=0$. We therefore have **Lemma** 2: Let m_0 be

$$m_0 = m - \text{rank}B. \tag{116}$$

Then the matrix A in Eq. (91) has $2m_0$ singular values with magnitude one.

5.5 Orthogonal matrices

In this paper integer matrices whose all rows (and all columns) are orthogonal each other play an important role. So we will describe them briefly.

Consider first the s-th order Hadamard matrix H_s. As an example H_4 is given as

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \tag{117}$$

The rows of H_4 are orthogonal each other, and the Euclidean norm of each row is $\sqrt{4}(=2)$. Therefore H_4 is not a so-called orthogonal matrix but is rewritten as

$$H_4 = \sqrt{4}\begin{bmatrix} \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ -\frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ -\frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \\ \frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & -\frac{1}{\sqrt{4}} & \frac{1}{\sqrt{4}} \end{bmatrix} \left(= 2\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) \tag{118}$$

In general we have

$$\|H_s\|_2 = \sqrt{s}, \quad \left(\text{i.e., } \frac{1}{\sqrt{s}}H_s \text{ is an orthogonal matrix}\right) \tag{119}$$

If a (real) matrix Θ is equal to a scalar multiple (κ) of an orthogonal matrix, we call it an "**e-orthogonal matrix with magnitude** κ". Here κ is not necessarily an integer as shown above and "e" is an abbreviation of "*extended*". Therefore by Eq. (119) H_s is an e-orthogonal matrix with magnitude $\sqrt{s}$. If Θ_1 and Θ_2 are e-orthogonal matrices with magnitude κ_1 and κ_2, then $\Theta_1\Theta_2$ is an e-orthogonal matrix with magnitude $\kappa_1\kappa_2$. If Θ_1 and Θ_2 are e-orthogonal and $\kappa_1=\kappa_2(=\kappa)$, then $\Theta_1 \oplus \Theta_2$ is an e-orthogonal matrix with magnitude κ. From these operations we can obtain a various e-orthogonal matrices. In the later applications it is desirable to generate a variety of e-orthogonal matrices with small magnitude.

We will show some examples of **integer e-orthogonal** matrices.

$$\Theta_1 = \begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = 5 \begin{bmatrix} 3/5 & 4/5 \\ -4/5 & 3/5 \end{bmatrix} (= 5U_1) \tag{120}$$

$$\Theta_2 = \begin{bmatrix} 3 & 4 & 0 \\ -4 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5 \begin{bmatrix} 3/5 & 4/5 & 0 \\ -4/5 & 3/5 & 0 \\ 0 & 0 & 5 \end{bmatrix} (= 5U_2) \tag{121}$$

$$\Theta_3 = \Theta_1\Theta_2, \ \text{etc} \tag{122}$$

$$\Theta_4 = \Theta_1 \oplus \Theta_2 \qquad (\text{provided that } \ \kappa_1 = \kappa_2) \tag{123}$$

Here U_1 and U_2 mean (real) unitary matrices.

An orthogonal matrix can be generated systematically from the well-known *Cayley transform* as follows: Let Q be a (real) *skew-symmetric matrix*, i.e., $Q^T = -Q$. Then

$$(\hat{U} \equiv)(\mathbf{1} + Q)(\mathbf{1} - Q)^{-1} \left(= (\mathbf{1} - Q)^{-1}(\mathbf{1} + Q)\right) \tag{124}$$

is an orthogonal matrix. If Q is an integer matrix, then $\hat{U}\det(\mathbf{1} - Q)$ is an integer e-orthogonal matrix but $\hat{U}$ is not.

Example 5**:** Let

$$Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \det(\mathbf{1} - Q) = \det\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\right) = 2$$

Then

$$\Theta = 2 \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

This is a rather trivial example.

Example 6**:** Let

$$Q = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad \det(\mathbf{1} - Q) = \det\left(\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}\right) = 5$$

Then

$$\Theta = (\mathbf{1} + Q)(\mathbf{1} - Q)^{-1}\det(\mathbf{1} - Q) = 5 \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \frac{1}{5} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = 5 \begin{bmatrix} -3/5 & 4/5 \\ -4/5 & -3/5 \end{bmatrix}$$

is an integer e-orthogonal matrix with magnitude 5.

5.6 Realization of Cases (i)–(iii) in Eqs. (88), (89) and (90)

5.6.1 Case (i) (2L Case)

The condition

$$\sigma_1 = \sigma_2 = \sigma_3 = \cdots = \sigma_m \ (\equiv \sigma_{max})$$

in Eq. (88) implies by Eqs. (114) and (111) that

$$\sigma_{B1} = \sigma_{B2} = \cdots = \sigma_{Bm}(\equiv \sigma_B) \tag{125}$$

Thus we have:

Lemma 3**:** Eq. (88) can be realized by the matrix in Eq. (91) if and only if B can be written as

$$B = \sigma_B U \quad (\sigma_B > 0; \quad U : \text{arbitrary real orthogonal matrix}) \tag{126}$$

i.e., B should be an e-orthogonal matrix. Of course $\sigma_B|u_{ij}| \le \mu$ have to be satisfied for Eq. (1). Note from Eqs. (88) and (125) that β in Eq. (14) is given as

$$\beta = 2m + m\sigma_B^2 \tag{127}$$

Therefore we have

$$\|B\|_2 = \sigma_B \tag{128}$$

$$\|A\|_2 = \sigma_1 = \sqrt{\frac{2+\sigma_{B1}^2+\sigma_{B1}\sqrt{\sigma_{B1}^2+4}}{2}} \approx \sigma_B \tag{129}$$

$$\|A^{-1}\|_2 = \sigma_{2m} = 1/\sigma_1 \tag{130}$$

$$\mathrm{Cond}_2(A) = \frac{\sigma_1}{\sigma_{2m}} = \sigma_1^2 \approx \sigma_{B1}^2 \tag{131}$$

Though B in Eq. (126) is a real matrix in general, an approximate integer matrix A can be easily obtained as Lemma 4.

Lemma 4: We can find a matrix A of the form in Eq. (91) such that its singular values satisfy Eq. (88). An algorithm to get an integer matrix from the prescribed $\mathrm{Cond}_2(A)$ is as follows:

(i) Determine $\sigma_B = \sigma_1 = \sqrt{\mathrm{Cond}_2(A)}$ by Eq. (131).

(ii) Let U be an integer e-orthogonal matrices with magnitude κ.

(iii) Determine B as

$$\Sigma = \frac{\sigma_B}{\kappa}\mathbf{1}, \quad B = UI(\Sigma) \tag{132}$$

Here $I(X)$ means an integer matrix obtained from the matrix X by rounding. The rounding produces some errors, but it is no problem for our application.

Since we can choose U arbitrarily, we can generate a variety of matrices A.

For your information we have

$$\sigma_B \le \|B\|_\infty < \sqrt{m}\sigma_B \tag{133}$$

$$\sigma_B + 1 \le \|A\|_\infty \le \sqrt{m}\sigma_B + 1 \tag{134}$$

$$(\sigma_B+1)^2 \le \mathrm{Cond}_\infty(A) \le (\sqrt{m}\sigma_B+1)^2 \tag{135}$$

cf. Eqs. (10) and (11)

On the other hand we have from Table I in Section 3:

$$\mathrm{Cond}_2(A) \le \frac{2\beta}{n} = \frac{2(2m+m\sigma_B^2)}{n} = 2+\sigma_B^2. \tag{136}$$

Comparing Eq. (131) with Eq. (133), we see that in this case Eq. (126) gives **nearly maximum condition number** for A in Eq. (91).

5.6.2 Case ii) (Logarithmic Case)

We will consider the realization of Eq. (89). Let the singular value decomposition os B as

$$B = U\Sigma V^T \quad (U \text{ and } V \text{ are orthogonal matrices}) \tag{137}$$

$$\Sigma = \mathrm{diag}\,[\sigma_{B1}, \sigma_{B2}, \cdots, \sigma_{Bm}], \quad \sigma_{B1} > \sigma_{B2} > \cdots > \sigma_{Bm} > 1 \tag{138}$$

From the first m equations of Eq. (89) and Eq. (111) we have

$$\begin{aligned} 2+\sigma_{B1}^2+\sigma_{B1}\sqrt{\sigma_{B1}^2+4} &= r^2\left(2+\sigma_{B2}^2+\sigma_{B2}\sqrt{\sigma_{B2}^2+4}\right) \\ &= r^4\left(2+\sigma_{B3}^2+\sigma_{B3}\sqrt{\sigma_{B3}^2+4}\right) \\ &= \cdots \\ &= r^{2(m-1)}\left(2+\sigma_{Bm}^2+\sigma_{Bm}\sqrt{\sigma_{Bm}^2+4}\right) \\ &= r^{2m}\left(2+\sigma_{Bm}^2-\sigma_{Bm}\sqrt{\sigma_{Bm}^2+4}\right) \end{aligned} \tag{139}$$

The last equation in Eq. (139) is derived from $\lambda_{m+} = r^2\lambda_{m-}$ (See Eq. (112). Equation (139) implies m equations, while there are $m+1$ variables (i.e., $\sigma_{Bi}(i=1,\cdots,m)$ and r). So one of these variables can be fixed and we can solve Eq. (139) for other m variables. Since

$$\mathrm{Cond}_2(A) = \frac{\sigma_1}{\sigma_{2m}} = \sigma_1^2 \tag{140}$$

and

$$\sigma_1 = \sqrt{\frac{2+\sigma_{B1}^2+\sigma_{B1}\sqrt{\sigma_{B1}^2+4}}{2}} \tag{141}$$

we regard σ_{B1} as a known and the others as unknowns.

For simplicity we assume in practical case that

$$\sigma_{Bi} > 2 \quad (i = 1, 2, \cdots, m) \tag{142}$$

Then

$$2+\sigma_{Bi}^2+\sigma_{Bi}\sqrt{\sigma_{Bi}^2+4} \approx 2\sigma_{Bi}^2 \tag{143}$$

holds.

The first $(m-1)$ equations of Eq. (139) are approximately written as follows:

$$2\sigma_{B1}^2 = 2r^2\sigma_{B2}^2 = 2r^4\sigma_{B3}^2 = 2r^6\sigma_{B4}^2 = \cdots = 2r^{2(m-1)}\sigma_{Bm}^2 \tag{144}$$

The last equation of Eq. (139) can be rewritten as

$$2+\sigma_{Bm}^2+\sigma_{Bm}\sqrt{\sigma_{Bm}^2+4} = r^2\left(2+\sigma_{Bm}^2-\sigma_{Bm}\sqrt{\sigma_{Bm}^2+4}\right) \tag{145}$$

from which we have

$$\begin{aligned} r^2 &= \frac{2+\sigma_{Bm}^2+\sigma_{Bm}\sqrt{\sigma_{Bm}^2+4}}{2+\sigma_{Bm}^2-\sigma_{Bm}\sqrt{\sigma_{Bm}^2+4}} \\ &= \frac{(2+\sigma_{Bm}^2+\sigma_{Bm}\sqrt{\sigma_{Bm}^2+4)^2(2+\sigma_{Bm}^2)^2-\sigma_{Bm}^2(\sigma_{Bm}^2+4)}}{} \\ &= \frac{(2+\sigma_{Bm}^2+\sigma_{Bm}\sqrt{\sigma_{Bm}^2+4)^2}}{4} \approx \sigma_{Bm}^4 \end{aligned} \tag{146}$$

We therefore have

$$r^2 \approx \sigma_{Bm}^4 \tag{147}$$

Substituting the above into Eqs. (144), we have

$$\sigma_{B1}^2 = \sigma_{Bm}^4 \cdot \sigma_{B2}^2 = \sigma_{Bm}^8 \cdot \sigma_{B3}^2 = \sigma_{Bm}^{12} \cdot \sigma_{B4}^2 = \cdots = \sigma_{Bm}^{4(m-1)} \cdot \sigma_{Bm}^2 \ \ (= \sigma_{Bm}^{4m-2}) \tag{148}$$

from which we have

$$\sigma_{Bm} = \sigma_{B1}^{\frac{1}{2m-1}} \tag{149}$$

$$\sigma_{B2} = \frac{\sigma_{B1}}{\sigma_{Bm}^2} = \frac{\sigma_{B1}}{\sigma_{B1}^{\frac{2}{2m-1}}} = \sigma_{B1}^{1-\frac{2}{2m-1}} = \sigma_{B1}^{\frac{2m-3}{2m-1}} \tag{150}$$

$$\sigma_{B3} = \frac{\sigma_{B1}}{\sigma_{Bm}^4} = \frac{\sigma_{B1}}{\sigma_{B1}^{\frac{4}{2m-1}}} = \sigma_{B1}^{1-\frac{4}{2m-1}} = \sigma_{B1}^{\frac{2m-5}{2m-1}} \tag{151}$$

Similarly we have in general

$$\sigma_{Bi} = \sigma_{B1}^{\frac{2m-2i+1}{2m-1}} \quad (i = 2, 3, \cdots, m-1, m) \tag{152}$$

Thus we can determine σ_{Bi} $(i = 2, \cdots, m)$ so that Eq. (89) holds approximately.

Lemma 5: We can find a matrix A of the form in Eq. (91) such that its singular values satisfy Eq. (89). An algorithm to get an integer matrix from the prescribed $\mathrm{Cond}_2(A)$ is as follows:

(i) Determine $\sigma_{B1} = \sigma_1 = \sqrt{\mathrm{Cond}_2(A)}$.

(ii) Determine σ_{Bi} from Eq. (152).

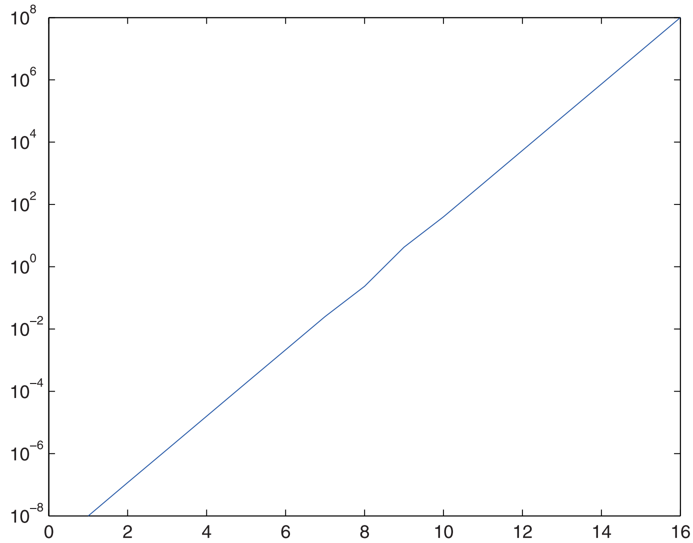

Fig. 1. Fig. 1 i-σ_i plot $(i = 1, 2, \cdots, 16)$ for Example 7.

(iii) Let U and V be an integer e-orthogonal matrices with magnitudes κ_1 and κ_2, respectively.
(iv) Determine B as

$$\Sigma = \frac{1}{\kappa_1\kappa_2}\text{diag}\,[\sigma_1, \sigma_2, \cdots, \sigma_m], \quad B = U\,(I(\Sigma))\,V^T \tag{153}$$

Since we can choose U and V arbitrarily, we can generate a variety of matrices A. Rounding errors due to $I(\Sigma)$ makes slight degradation from Eq. (89), but it is no problem for our application.
Example 7: Let $n = 16$, $\kappa_1 = \kappa_2 = 2\sqrt{2}$, and $\text{Cond}_2(A) = 10^{16}$.

So we have $\sigma_1 = \sqrt{\text{Cond}_2(A)} = 10^8$. Using Eq. (149), we find σ_{Bi} and then B and A. Then the singular values of A are as follows:

$$\sigma_1 = 1.0 \cdot 10^8,\ \sigma_2 = 8.57696 \cdot 10^6,\ \sigma_3 = 7.35642 \cdot 10^7,\ \sigma_4 = 6.3096 \cdot 10^4,\ \sigma_5 = 5.4117 \cdot 10^3,$$
$$\sigma_6 = 4.6416 \cdot 10^2,\ \sigma_7 = 3.9811 \cdot 10,\ \sigma_8 = 3.4145,\ \ \sigma_k = 1/\sigma_{n-k+1}(k = m+1, \cdots, n)$$

The relation i and σ_i are plotted in Fig. 1. As seen, the singular values distributed in very good linearity. Note that the central part of the curve deviates slightly from the linearity. This is because the rounding error is large when σ_{Bi} is small. The condition number of A is given as $1.0 \cdot 10^{16}$, as is expected.

5.6.3 Case iii)(3L Case)

In order to realize 3L Case (Eq. (90), we set

$$B = G_1 D G_2^T \tag{154}$$

where G_1 and G_2 are $m \times l$ submatrices composed of the first l columns of any e-orthogonal matrices with magnitude κ_1 and κ_2 and D is a positive definite integer diagonal matrix, i.e.,

$$D = \text{diag}\,[d_1, d_2, \cdots, d_l] \tag{155}$$

If we suppose that

$$d_1 = d_2 = \cdots = d_l \tag{156}$$

then we can realize Eq. (90).

If we choose d_i as arbitrary positive values, then

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_l \gg 1, \quad \sigma_{l+1} = \cdots = \sigma_{n-l} = 1, \quad 1 \gg \sigma_{n-l+1} \geq \cdots \geq \sigma_n \tag{157}$$

can be realized.

6. Conclusions

This paper studies the generation of an integer ill-conditioned matrix $A = [a_{ij}]$ with $|a_{ij}| \leq \mu$. First some upper bounds of the condition number of A are shown in terms of μ, $\Sigma_{i,j} a_{ij}^2$, and n.

Then an innovative generation method for extremely ill-conditioned integer matrices ia shown. This method is superior to the original Rump's method in some respects. i.e., the former (i) has a simpler algorithm, and (ii) can generate more variety of ill-conditioned matrices than the latter.

The method as well as the previous ones [7–10] has a serious drawback. That is, the inverse A^{-1} is very close to a matrix with rank one. So finally in order to solve the above problems we propose a desirable singular value distribution of ill-conditioned benchmark matrices and give partial solutions to them.

Some of the upper bounds given in this paper seem much overestimated. So more tight upper bounds shall be further studied in the future.

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