Conservatism of the Circle Criterion - Solution of a Problem posed by A. Megretski

Siegfried M. Rump

Abstract — In the collection of open problems in mathematical systems and control theory [1] Alexandre Megretski posed a problem from which it follows how conservative the well-known circle criterion may be. We solve this problem.

Keywords — circle criterion, robust stabilization, Perron-Frobenius

In [8] Alexandre Megretski posed a problem 1 (Problem 30) with certain implications: in harmonic analysis a connection between the time domain and frequency domain multiplications, in control theory the conservatism of the circle criterion, the possibility of robust stabilization of a second-order uncertain system using a linear and time-invariant controller, and the conjectured finiteness of the gap between the minimum in some specially structured non-convex quadratic optimization problem and its natural relaxation (cf. [1, Problem 30]). Part 3 of the posed problem is as follows (\(\sigma_{\text{max}}\) denotes the largest singular value).

**PROBLEM.** Does there exist a finite constant \(\gamma > 0\) with the following feature: for any cyclic \(n\times n\) real matrix \(2\)

\[H = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_{n-1} & h_0 & \cdots & h_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ h_1 & h_2 & \cdots & h_0 \end{bmatrix}, \tag{1}\]

such that \(\sigma_{\text{max}}(H) \geq \gamma\), there exists a non-zero real vector \(x\) such that \(|y_i| \geq |x_i|\) for all \(i = 0, 1, \ldots, n - 1\) where

\[x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}, \quad y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = Hx. \]

As Megretski mentions, the solution of this problem implies positive answers to the other two subproblems posed under problem number 30 in [1]. We solve the problem in the affirmative by giving narrow bounds for \(\gamma\) depending on the dimension of the matrix. We prove the following theorem.

**Theorem 1:** For any matrix \(H \in \mathbb{R}^{n \times n}\) of the form (1) with

\[\sigma_{\text{max}}(H) \geq (3 + 2\sqrt{2}) \cdot n\]

there exists some nonzero \(x \in \mathbb{R}^n\) with

\[|(Hx)_\nu| \geq |x_\nu| \quad \text{for } 1 \leq \nu \leq n. \tag{2}\]

Furthermore, there exists a sequence of matrices \(H(n) \in \mathbb{R}^{n \times n}, 1 \leq n \in \mathbb{N}\), with

\[\sigma_{\text{max}}(H(n)) \geq \frac{1}{2}n \quad \text{for } n \text{ odd}
\]

and

\[\sigma_{\text{max}}(H(n)) \geq \frac{1}{4}n \quad \text{for } n \text{ even},\]

such that for all \(n \in \mathbb{N}\) there does not exist a nonzero vector \(x \in \mathbb{R}^n\) with (2).

For the solution of Megretski’s problem we need the extension of classical Perron-Frobenius theory from nonnegative to arbitrary real matrices [10]. This theory was developed to solve (cf. [11]) the conjecture that the component-wise distance to the nearest singular matrix is proportional to the reciprocal of its (componentwise) condition number [3, p.18], [5, p.140].

For a real matrix \(A \in \mathbb{R}^{n \times n}\) define the **real spectral radius** [9] to be

\[\rho_0(A) := \max\{|\lambda| : \lambda \text{ real eigenvalue of } A\}\]

and \(\rho_0(A) := 0\) if \(A\) has no real eigenvalue. The set of signature matrices is defined by

\[\{S \in \mathbb{R}^{n \times n} : S \text{ diagonal with } |S_{ii}| = 1 \text{ for } 1 \leq i \leq n\}\]

Throughout the paper we will use absolute value of vectors and matrices and comparison of those **componentwise**. Thus, for example, \(\{D \in \mathbb{R}^{n \times n} : D \text{ diagonal with } |D| \leq I\}\) consists of all diagonal matrices with \(-1 \leq D_{ii} \leq 1\) for \(i \in \{1, \ldots, n\}\).

The **sign-real spectral radius** [10] is defined by

\[\rho_0^S(A) := \max_{|S|=I} \rho_0(\tilde{S}A). \tag{3}\]

It maximizes the real spectral radius when multiplying the rows of \(A\) independently by \(\pm 1\). This quantity generalizes many properties of the Perron root \(\rho(A)\) of nonnegative matrices to general real matrices. Among the characterizations we need are the following.

**Theorem 2:** For \(A \in \mathbb{R}^{n \times n}\) the following is true:

i) \(\rho_0^S(A) = \min\{0 \leq r \in \mathbb{R} : \det((rI - SA) \geq 0 \text{ for all } |S| = I}\).

ii) For \(0 \leq r \in \mathbb{R}\) and \(\det((rI - A) \neq 0\) it is

\[\rho_0^S(A) < r \iff (rI - A)^{-1}(rI + A) \in \mathcal{P},\]

where \(\mathcal{P}\) denotes the class of matrices with all principal minors positive.

iii) \(\rho_0^S(A) = \max_{0 \neq x \in \mathbb{R}^n \neq 0} \frac{|Ax|}{x_i}\).

iv) For \(0 \leq r \in \mathbb{R}\) it is

\[\rho_0^S(A) \geq r \iff \exists 0 \neq x \in \mathbb{R}^n : |Ax| \geq r|x|\].
The parts are proven in [10, Theorems 2.3, 2.13, 3.1] and iv) is a consequence of iii). Part iv) gives a simple way to compute lower bounds of $\rho_0^S(A)$ for a given matrix $A$; upper bounds are difficult, in fact NP-hard to calculate [10, Theorem 3.5, Corollary 2.9]. The key to the solution of the PROBLEM are lower bounds for $\rho_0^S$ depending on the geometric mean of cycles. Given a cycle $\mu = (\mu_1, \ldots, \mu_k) \subseteq \{1, \ldots, n\}$, $1 \leq |\mu| := k \leq n$, it is

$$\prod A_{\mu} = |A_{\mu_1 \mu_2} \cdots A_{\mu_{k-1} \mu_k} A_{\mu_k \mu_1}|^{1/k}. $$

Note that the diagonal elements of $A$ form cycles of length $1$.

**Theorem 1.** Given a circulant $\rho$ where circulants have a number of interesting properties [2], among the bounds for $\rho_0$ are [10, Theorem 3.5, Corollary 2.9].

**Proof:** [11, Theorem 4.4] \hfill \blacksquare

These results give the key to solve the PROBLEM. For the solution we need some more notation. A matrix of type (1) are is called *circulant* in matrix theory [6]. Denoting the permutation matrix $P \in \mathbb{R}^{n \times n}$ with $p_{12} = \cdots = p_{n-1,n} = p_{n1} = 1$ it is

$$H = \begin{pmatrix} h_0 & h_1 & h_2 & \cdots & h_{n-1} \\ h_{n-1} & h_0 & h_1 & \cdots & h_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1 & h_2 & h_3 & \cdots & h_0 \end{pmatrix} $$

$$= \sum_{\nu=0}^{n-1} h_\nu P^{\nu} \in \mathbb{R}^{n \times n}. $$

Note that indices of $h$ are running form 0 to $n - 1$. Circulants have a number of interesting properties [2], among them that circulants are normal, i.e. $H = Q\Lambda Q^*$ for unitary $Q \in \mathbb{C}^{n \times n}$ and diagonal $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n)$. The eigenvalues of *every* circulant $H$ can be ordered such that

$$Q := -n^{-1/2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \omega & \omega^2 & \omega^3 & \omega^4 \\ \omega^2 & \omega^4 & \omega^6 & \omega^8 \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{n-1} & \omega^{2n-1} & \omega^{3n-1} & \omega^{4n-1} \end{pmatrix} $$

diagonalizes $H$, where $\omega = e^{2\pi i/n}$. Hence for every circulant $H$,

$$\sigma_{\text{max}}(H) = ||H||_2 = ||\Lambda||_2 = \rho(H) $$

where $\rho$ denotes the spectral radius.

With these preliminaries we can prove the first part of Theorem 1. Given a circulant $H$ as in (1) with $||H||_2 \geq (3 + 2\sqrt{2})n$, it follows by Perron-Frobenius theory [12, Theorem 2.8]

$$(3 + 2\sqrt{2})n \leq ||H||_2 = \rho(H) \leq \rho(||H||)$$

$$= \sum_{\nu=0}^{n-1} |h_\nu| \leq n \cdot \max_{0 \leq \nu \leq n-1} |h_\nu|, $$

The diagonals form cycles with geometric mean $|h_\nu|$, and by (7), $\max |h_\nu| \geq 3 + 2\sqrt{2}$. Hence, Theorem 3 implies $\rho_0^S(H) \geq 1$, and Theorem 2, iv) proves the first part of Theorem 1.

To prove the second part define

$$H(n) := \begin{pmatrix} \text{circ}(0,1,1, \ldots, 1,-1,-1, \ldots, -1) \text{ for } n \text{ odd} \\ \text{circ}(0,1,1,1,0,1,0, \ldots, -1,1,0, \ldots, -1) \text{ for } n \text{ even} \end{pmatrix} $$

\begin{equation}
\rho_0(H) = n^{-1/2} \sum_{\nu=0}^{n-1} h_\nu \omega^{\nu}, \quad \omega = e^{2\pi i/n},
\end{equation}

with orthonormal eigenvector matrix $Q$ as in (5). The matrices $H = H(n)$ as defined in (8) are skew-symmetric for every $n$. Thus eigenvalues are purely imaginary and

$$||H||_2 = \rho(H) = \sum_{\nu=0}^{n-1} |\omega^{\nu}| $$

$$= 2 \cdot \Im \sum_{\nu=0}^{\lfloor n/2 \rfloor} \omega^{\nu} $$

For even dimension $n$ it is $\sum_{\nu=0}^{n/2} \omega^{\nu} = \omega^{n/2} = \omega^{2n} = 2$ because $\omega^{n/2} = 0$. For odd dimension we proceed similarly and a computation yields

$$||H(n)||_2 = \begin{cases} 2 \cdot \cot \frac{\pi}{n} \cot \frac{\pi}{n} + \sin \frac{\pi}{n} \text{ for } n \text{ even} \\ 2 \cdot \cot \frac{\pi}{n} \text{ for } n \text{ odd} \end{cases} $$

In any case one verifies

$$||H(n)||_2 \geq 2 \cdot \cot \frac{\pi}{n} \geq \frac{n}{2} $$

(10)

To proceed further we need a slightly different upper bound for $\rho_0^S$ which can be proven using Theorem 2, ii) and a continuity argument. We choose to give a different (from [10]) and simple proof of the following. A similar argument has been used in [7].

**Lemma 4.** Let $A \in \mathbb{R}^{n \times n}$ and $0 < r \in \mathbb{R}$ be given. If $rI - A$ is nonsingular and all minors of the Cayley transform

$$C = (rI - A)^{-1} (rI + A) $$

are nonnegative, then $\rho_0^S(A) \leq r$.

**Proof:** With $C \in P_0$, the class of matrices with all minors nonnegative, it is $C(I - D) \in P_0$ for every diagonal $D$ with $0 \leq D \leq I$, and also $C(I - D) + D \in P_0$ (by expanding the determinant, see also [4, Theorem 5.26]). It is

$$C(I - D) + D = (rI - A)^{-1} (rI + A - 2AD). $$

For $D = \frac{1}{2} I$ and $rI - A$ being nonsingular it follows $\det(rI - A)^{-1} > 0$, and using all possibilities $|D| = I$ it follows $\det(rI - AS) = \det(rI - SA) \geq 0$ for all $|S| = I$. Theorem 2, ii) finishes the proof. \hfill \blacksquare
For $n$ odd and $k = (n-1)/2$ the eigenvalues of $H = H(n)$ compute to
\[
\lambda_m(H) = \frac{1 - \omega^{(k+1)m}}{1 + \omega^{(k+1)m}} \quad \text{for } 0 \leq m \leq n - 1.
\]
Therefore, the eigenvalues of the Cayley transform $(I - H)^{-1}(I + H)$ are the roots of unity, and a computation shows that it also holds for $n$ even. This is straightforward for $1$ is proved.

Replacing $H$ by $H + I$ produces matrices $H$ with $\rho_0^S(H) = 1$ and $\sigma_{\max}(H) \geq \frac{n}{2}$ for $n$ odd, and $\sigma_{\max}(H) \geq \frac{n}{2}$ for $n$ even. This proves Theorem 1 for $n \geq 4$. A simple computation shows that it also holds for $n \leq 3$. Theorem 1 is proved.

Finally, we remark that
\[
\rho_0^S(A) = \|H\| \quad \text{for a circulant } H \text{ and } n \in \{1, 2, 4\}.
\]
This is straightforward for $n \in \{1, 2\}$ using the characterizations given before, and for $n = 4$, $H = \text{circ}(h_0, h_1, h_2, h_3)$ and using (9) one can show that
\[
\|H\| = \rho(H) = \max \left( \left| (h_1 - h_3) + i(h_0 - h_2) \right|, \left| h_0 - h_1 + h_2 - h_3 \right|, \left| \sum_{\nu=0}^{3} h_\nu \right| \right).
\]
Choosing suitable signature matrices $S$ shows $\rho_0^S(H) = \rho(H) \equiv \|H\|$. This implies

**Corollary 5:** For a circulant $H \in \mathbb{R}^{n \times n}$, $n \in \{1, 2, 4\}$ it is
\[
\|H\| \geq 1 \iff \exists x \neq 0 \in \mathbb{R}^n : \|Hx\| \geq |x|.
\]

**REFERENCES**


