Conservatism of the Circle Criterion - Solution of a Problem posed by A. Megretski

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Abstract— **In the collection of open problems in mathematical systems and control theory [1] Alexandre Megretski posed a problem from which it follows how conservative the well-known circle criterion may be. We solve this problem.**

Keywords— **circle criterion, robust stabilization, Perron-Frobenius**

In [8] Alexandre Megretski posed a problem [1] (Problem 30) with certain implications: in harmonic analysis a connection between the time domain and frequency domain multiplications, in control theory the conservatism of the circle criterion, the possiblility of robust stabilization of a second-order uncertain system using a linear and time-invariant controller, and the conjectured finiteness of the gap between the minimum in some specially structured non-convex quadratic optimization problem and its natural relaxation (cf. [1, Problem 30]). Part 3 of the posed problem is as follows ($\sigma_{\max}$ denotes the largest singular value).

PROBLEM. *Does there exist a finite constant $\gamma > 0$ with the following feature: for any cyclic n-by-n real matrix* [2]

$$H = \begin{bmatrix} h_0 & h_1 & \dots & h_{n-1} \\ h_{n-1} & h_0 & \dots & h_{n-2} \\ & & \vdots & \\ h_1 & h_2 & \dots & h_0 \end{bmatrix} \tag{1}$$

such that $\sigma_{\max}(H) \geq \gamma$, there exists a non-zero real vector x such that $|y_i| \geq |x_i|$ for all $i = 0, 1, \dots, n-1$ where

$$x = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix}, y = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{bmatrix} = Hx.$$

As Megretski mentions, the solution of this problem implies positive answers to the other two subproblems posed under problem number 30 in [1]. We solve the problem in the affirmative by giving narrow bounds for γ depending on the dimension of the matrix. We prove the following theorem.

Theorem 1: For any matrix $H \in \mathbf{R}^{n\times n}$ of the form (1) with

$$\sigma_{\max}(H) \geq (3 + 2\sqrt{2}) \cdot n$$

there exists some nonzero $x \in \mathbf{R}^n$ with

$$|(Hx)_\nu| \geq |x_\nu| \quad \text{for } 1 \leq \nu \leq n. \tag{2}$$

Furthermore, there exists a sequence of matrices $H_{(n)} \in \mathbf{R}^{n\times n}, 1 \leq n \in \mathbf{N}$, with

$$\sigma_{\max}(H_{(n)}) \geq \tfrac{1}{2}n \quad \text{for } n \text{ odd}$$

and

$$\sigma_{\max}(H_{(n)}) \geq \tfrac{1}{4}n \quad \text{for } n \text{ even},$$

such that for all $n \in \mathbf{N}$ there does not exist a nonzero vector $x \in \mathbf{R}^n$ with (2).

For the solution of Megretski's problem we need the extension of classical Perron-Frobenius theory from nonnegative to arbitrary real matrices [10]. This theory was developed to solve (cf. [11]) the conjecture that the componentwise distance to the nearest singular matrix is proportional to the reciprocal of its (componentwise) condition number [3, p.18], [5, p.140].

For a real matrix $A \in \mathbf{R}^{n\times n}$ define the *real spectral radius* [9] to be

$$\rho_0(A) := \max\{|\lambda| : \lambda \text{ real eigenvalue of } A\}$$

and $\rho_0(A) := 0$ if A has no real eigenvalue. The set of signature matrices is defined by

$$\{S \in \mathbf{R}^{n\times n} : S \text{ diagonal with } |S_{ii}| = 1 \text{ for } 1 \leq i \leq n\}.$$

Throughout the paper we will use absolute value of vectors and matrices and comparison of those *componentwise.* Thus, for example, $\{D \in \mathbf{R}^{n\times n} : D \text{ diagonal with } |D| \leq I\}$ consists of all diagonal matrices with $-1 \leq D_{ii} \leq 1$ for $i \in \{1, \dots, n\}$.

The *sign-real spectral radius* [10] is defined by

$$\rho_0^S(A) := \max_{|\tilde{S}|=I} \rho_0(\tilde{S}A). \tag{3}$$

It maximizes the real spectral radius when multiplying the rows of A independently by ± 1. This quantity generalizes many properties of the Perron root $\rho(A)$ of nonnegative matrices to general real matrices. Among the characterizations we need are the following.

Theorem 2: For $A \in \mathbf{R}^{n\times n}$ the following is true:

i) $\rho_0^S(A) = \min\{0 \leq r \in \mathbf{R} : \det(rI - SA) \geq 0 \text{ for all } |S| = I\}$.

ii) For $0 \leq r \in \mathbf{R}$ and $\det(rI - A) \neq 0$ it is

$$\rho_0^S(A) < r \quad \Leftrightarrow \quad (rI - A)^{-1}(rI + A) \in \mathcal{P},$$

where $\mathcal{P}$ denotes the class of matrices with all principal minors positive.

iii) $\rho_0^S(A) = \max_{0\neq x\in\mathbf{R}^n} \min_{x_i\neq 0} \left|\frac{(Ax)_i}{x_i}\right|$.

iv) For $0 \leq r \in \mathbf{R}$ it is

$$\rho_0^S(A) \geq r \quad \Leftrightarrow \quad \exists\, 0 \neq x \in \mathbf{R}^n : |Ax| \geq r|x|.$$

[1] The author wishes to thank P. Batra for pointing to this problem.
[2] Note indices run from 0 to $n-1$.

The parts are proven in [10, Theorems 2.3, 2.13, 3.1] and *iv)* is a consequence of *iii)*. Part *iv)* gives a simple way to compute lower bounds of $\rho_0^S(A)$ for a given matrix A; upper bounds are difficult, in fact NP-hard to calculate [10, Theorem 3.5, Corollary 2.9].

The key to the solution of the PROBLEM are lower bounds for ρ_0^S depending on the geometric mean of cycles. Given a cycle $\mu = (\mu_1, \ldots, \mu_k) \subseteq \{1, \ldots, n\}, 1 \le |\mu| := k \le n$, it is

$$|\prod A_\mu|^{1/|\mu|} = |A_{\mu_1\mu_2} \cdot \ldots \cdot A_{\mu_{k-1}\mu_k} \cdot A_{\mu_k\mu_1}|^{1/k}.$$

Note that the diagonal elements of A form cycles of length 1.

Theorem 3: For $A \in \mathbf{R}^{n\times n}$ and a cycle $\mu \subseteq \{1, \ldots, n\}$ it is

$$\rho_0^S(A) \ge (3 + 2\sqrt{2})^{-1} \cdot |\prod A_\mu|^{1/|\mu|}.$$

Proof: [11, Theorem 4.4] ∎

These results give the key to solve the PROBLEM. For the solution we need some more notation. A matrix of type (1) are is called *circulant* in matrix theory [6]. Denoting the permutation matrix $P \in \mathbf{R}^{n\times n}$ with $p_{12} = \ldots = p_{n-1,n} = p_{n1} = 1$ it is

$$\begin{aligned} H &= \operatorname{circ}(h_0, \ldots, h_{n-1}) \\ &= \begin{pmatrix} h_0 & h_1 & h_2 & \ldots & h_{n-1} \\ h_{n-1} & h_0 & h_1 & \ldots & h_{n-2} \\ \ddots & \ddots & \ddots & & \\ h_1 & h_2 & h_3 & \ldots & h_0 \end{pmatrix} \\ &= \sum_{\nu=0}^{n-1} h_\nu P^\nu \in \mathbf{R}^{n\times n}. \end{aligned} \tag{4}$$

Note that indices of h are running form 0 to $n-1$. Circulants have a number of interesting properties [2], among them that circulants are normal, i.e. $H = Q\Lambda Q^*$ for unitary $Q \in \mathbf{C}^{n\times n}$ and diagonal $\Lambda = \operatorname{diag}(\lambda_1, \ldots \lambda_n)$. The eigenvalues of *every* circulant H can be ordered such that

$$Q := n^{-1/2} \cdot \begin{pmatrix} 1 & 1 & 1 & & 1 \\ 1 & \omega & \omega^2 & & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \ldots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & & \omega^{(n-1)(n-1)} \end{pmatrix} \tag{5}$$

diagonalizes H, where $\omega = e^{2\pi i/n}$. Hence for every circulant H,

$$\sigma_{\max}(H) = \|H\|_2 = \|\Lambda\|_2 = \rho(H) \tag{6}$$

where ρ denotes the spectral radius.

With these preliminaries we can prove the first part of Theorem 1. Given a circulant H as in (1) with $\|H\|_2 \ge (3 + 2\sqrt{2})n$, it follows by Perron-Frobenius theory [12, Theorem 2.8]

$$\begin{aligned} (3 + 2\sqrt{2})n \le \|H\|_2 &= \rho(H) \le \rho(|H|) \\ &= \sum_{\nu=0}^{n-1} |h_\nu| \le n \cdot \max_{0\le\nu\le n-1} |h_\nu|. \end{aligned} \tag{7}$$

The diagonals form cycles with geometric mean $|h_\nu|$, and by (7), $\max |h_\nu| \ge 3 + 2\sqrt{2}$. Hence, Theorem 3 implies $\rho_0^S(H) \ge 1$, and Theorem 2, *iv)* proves the first part of Theorem 1.

To prove the second part define

$$H_{(n)} := \begin{cases} \operatorname{circ}(0, 1, 1, \ldots, 1, -1, -1, \ldots, -1) & \text{for } n \text{ odd} \\ \operatorname{circ}(0, 1, 1, \ldots, 1, 0, -1, -1, \ldots, -1) & \text{for } n \text{ even.} \end{cases} \tag{8}$$

The first row of $H_{(n)}$ comprises of an equal number of $k := \lfloor (n-1)/2 \rfloor$ components $+1$ and -1. The eigenvalues $\lambda_m(H)$ of a circulant $H = \operatorname{circ}(h_0, \ldots, h_{n-1}) \in \mathbf{R}^{n\times n}$ are [2]

$$\lambda_m(H) = \sum_{\nu=0}^{n-1} h_\nu \omega^{m\nu}, \quad \omega = e^{2\pi i/n}, \tag{9}$$

with orthonormal eigenvector matrix Q as in (5). The matrices $H = H_{(n)}$ as defined in (8) are skew-symmetric for every n. Thus eigenvalues are purely imaginary and

$$\left.\begin{aligned} \|H\|_2 &= \rho(H) = \left|\sum_{\nu=0}^{n-1} \omega^\nu\right| \\ &= 2 \cdot \mathcal{I}m \sum_{\nu=0}^{\lfloor n/2 \rfloor} \omega^\nu \end{aligned}\right. \quad \text{for every } n \in \mathbf{N}.$$

For even dimension n it is $\sum_{\nu=0}^{n/2} \omega^\nu = \frac{\omega^{n/2}-1}{\omega-1} = \frac{-2}{\omega-1}$ because $\omega^{n/2} = 0$. For odd dimension we proceed similarly and a computation yields

$$\|H_{(n)}\|_2 = \begin{cases} 2 \cdot \cot \frac{\pi}{n} & \text{for } n \text{ even} \\ (1 + \cos \frac{\pi}{n}) \cot \frac{\pi}{n} + \sin \frac{\pi}{n} & \text{for } n \text{ odd.} \end{cases}$$

In any case one verifies

$$\|H_{(n)}\|_2 \ge 2 \cdot \cot \frac{\pi}{n} \ge \frac{n}{2} \quad \text{for } n \ge 4. \tag{10}$$

To proceed further we need a slightly different upper bound for ρ_0^S which can be proven using Theorem 2, *ii)* and a continuity argument. We choose to give a different (from [10]) and simple proof of the following. A similar argument has been used in [7].

Lemma 4: Let $A \in \mathbf{R}^{n\times n}$ and $0 < r \in \mathbf{R}$ be given. If $rI - A$ is nonsingular and all minors of the Cayley transform

$$C = (rI - A)^{-1}(rI + A)$$

are nonnegative, then $\rho_0^S(A) \le r$.

Proof: With $C \in \mathcal{P}_0$, the class of matrices with all minors nonnegative, it is $C \cdot (I - D) \in \mathcal{P}_0$ for every diagonal D with $0 \le D \le I$, and also $C(I - D) + D \in \mathcal{P}_0$ (by expanding the determinant, see also [4, Theorem 5.26]). It is

$$C(I - D) + D = (rI - A)^{-1}(rI + A - 2AD).$$

For $D = \frac{1}{2}I$ and $rI - A$ being nonsingular it follows $\det(rI - A)^{-1} > 0$, and using all possiblilities $|D| = I$ it follows $\det(rI - AS) = \det(rI - SA) \ge 0$ for all $|S| = I$. Theorem 2, *i)* finishes the proof. ∎

For n odd and $k = (n-1)/2$ the eigenvalues of $H = H_{(n)}$ compute to

$$\lambda_m(H) = \frac{1-\omega^{(k-1)m}}{1+\omega^{(k-1)m}} \quad \text{for } 0 \le m \le n-1.$$

Therefore, the eigenvalues of the Cayley transform $(I - H)^{-1}(I + H)$ are the roots of unity, and a computation yields

$$(I-H)^{-1}(I+H) = Q \cdot \text{diag}(\omega^{-(k+1)m})_{0\le m\le n-1} \cdot Q^* = P^k$$

with P being the permutation matrix $\text{circ}(0, 1, 0, \ldots, 0)$. Because n is odd, every minor of P and of every power of P is nonnegative. Thus Lemma 4 shows $\rho_0^S(H) \le 1$. By $|Hx| \ge |x|$ for $x = (1, 1, 0, \ldots, 0)^T$ and Theorem 2, *iv)* it follows $\rho_0^S(H_{(n)}) = 1$ for n odd.

For n even things are a little more complicated. One can show

$$C := (2I - H)^{-1}(2I + H) = \frac{1}{2}\text{circ}(1, z, 1, 1, z, -1)$$

where z is a row vector of $\frac{n}{2} - 1$ zeros. Some more involved computation shows that all minors of C are nonnegative and Lemma 4 implies $\rho_0^S(H) \le 2$. For the signature matrix S with diagonal element $S_{\nu\nu} = -1$ for $\nu \in \{1, \frac{n}{2} + 1\}$, and $+1$ otherwise it is $\det(2I - SH) = 0$, and by Theorem 2, *i)* it follows $\rho_0^S(H_{(n)}) = 2$ for n even. Summarizing

$$\rho_0^S(H_{(n)}) = \begin{cases} 1 & \text{for } n \text{ odd} \\ 2 & \text{for } n \text{ even.} \end{cases} \tag{11}$$

Replacing $H_{(2n)}$ by $\frac{1}{2}H_{(2n)}$ produces matrices H with $\rho_0^S(H) = 1$ and $\sigma_{\max}(H) \ge \frac{n}{2}$ for n odd, and $\sigma_{\max}(H) \ge \frac{n}{4}$ for n even. This proves Theorem 1 for $n \ge 4$. A simple computation shows that it also holds for $n \le 3$. Theorem 1 is proved.

Finally we remark that

$$\rho_0^S(A) = \|H\| \quad \text{for a circulant } H \text{ and } n \in \{1, 2, 4\}.$$

This is straightforward for $n \in \{1, 2\}$ using the characterizations given before, and for $n = 4$, $H = \text{circ}(h_0, h_1, h_2, h_3)$ and using (9) one can show that

$$\|H\| = \rho(H) = \max\left(|(h_1 - h_3) + i(h_0 - h_2)|, \; |h_0 - h_1 + h_2 - h_3|, \left|\sum_{\nu=0}^{3} h_\nu\right| \right).$$

Choosing suitable signature matrices S shows $\rho_0^S(H) = \rho(H) = \|H\|$. This implies

Corollary 5: For a circulant $H \in \mathbf{R}^{n\times n}$, $n \in \{1, 2, 4\}$ it is

$$\|H\| \ge 1 \quad \Leftrightarrow \quad \exists\, 0 \ne x \in \mathbf{R}^n : |Hx| \ge |x|.$$

References

[1] V.D. Blondel, E.D. Sontag, M. Vidyasagar, and J.C. Willems. *Open Problems in Mathematical Systems and Control Theory.* Springer, 1999.

[2] P.J. Davis. *Circulant Matrices.* John Wiley, New York, 1979.

[3] J.W. Demmel. The Componentwise Distance to the Nearest Singular Matrix. *SIAM J. Matrix Anal. Appl.*, 13(1):10–19, 1992.

[4] M. Fiedler. *Special matrices and their applications in numerical mathematics.* Martinus Nijhoff Publishers, Dordrecht, 1986.

[5] N.J. Higham. *Accuracy and Stability of Numerical Algorithms.* SIAM Publications, Philadelphia, 1996.

[6] R.A. Horn and Ch. Johnson. *Matrix Analysis.* Cambridge University Press, 1985.

[7] C.R. Johnson and M.J. Tsatsomeros. Convex Sets of Nonsingular and P-Matrices. *LAMA*, 38(3):233–239, 1995.

[8] A. Megretski. Problem 30: How conservative is the circle criterion? In V.D. Blondel, E.D. Sontag, M. Vidyasagar, and J.C. Willems, editors, *Open Problems in Mathematical Systems and Control Theory*, pages 149–151. Springer, London, 1999.

[9] J. Rohn. Systems of Linear Interval Equations. *Linear Algebra Appl. 126*, pages 39–78, 1989.

[10] S.M. Rump. Theorems of Perron-Frobenius type for matrices without sign restrictions. *Linear Algebra and its Applications (LAA)*, 266:1–42, 1997.

[11] S.M. Rump. Ill-conditioned Matrices are componentwise near to singularity. *SIAM Review (SIREV)*, 41(1):102–112, 1999.

[12] R.S. Varga. *Matrix Iterative Analysis.* Prentice-Hall, Englewood Cliffs, N.J., 1962.