# ESTIMATES OF THE DETERMINANT OF A PERTURBED IDENTITY MATRIX*

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**Abstract.** Recently Brent et al. presented new estimates for the determinant of a real perturbation $I + E$ of the identity matrix. They give a lower and an upper bound depending on the maximum absolute value of the diagonal and the off-diagonal elements of $E$, and show that either bound is sharp. Their bounds will always include 1, and the difference of the bounds is at least $\mathrm{tr}(E)$. In this note we present a lower and an upper bound depending on the trace and Frobenius norm $\epsilon := \|E\|_F$ of the (real or complex) perturbation $E$, where the difference of the bounds is not larger than $\epsilon^2 + \mathcal{O}(\epsilon^3)$ provided $\epsilon < 1$.

**Key words.** Determinant, Hadamard bound, Ostrowski bound, perturbation of identity

**AMS subject classifications.** 15A15, 65F40

**1. Main result.** Classical estimates for the determinant of a matrix include the Hadamard bound [7] or Gershgorin circles [6]. Moreover, Ostrowski [11, 12, 13] gave a number of lower and upper bounds. Other estimates include [4, 9, 1]. In particular, bounds for the determinant of a perturbed identity matrix are given in Ostrowski's papers or [15].

Recently, new bounds for $\det(I+E)$ have been presented by Brent et al. in [2, 3]. Denote by $\delta$ the maximum absolute value of the diagonal elements, and by $\varepsilon$ the maximum absolute value of the off-diagonal elements of $E$. Then [2, 3] proves

$$(1-\delta-(n-1)\varepsilon)(1-\delta+\varepsilon)^{n-1} \le \det(I+E) \le ((1+\delta)^2+(n-1)\varepsilon^2)^{n/2}, \tag{1.1}$$

where $\delta + (n-1)\varepsilon \le 1$ is supposed for the left inequality. Both inequalities is sharp as by explicit examples given in [2, 3]. For convergent $E$, Fredholm's identity [5]

$$\det(I+E) = \exp\left(\sum_{k=1}^{n}(-1)^{k-1}\frac{\mathrm{tr}(E^k)}{k}\right) \tag{1.2}$$

yields $\det(I+E) = \exp(\mathrm{tr}(E)) + \mathcal{O}(\varepsilon^2)$ for $\|E\| \le \varepsilon < 1$ and some matrix norm $\|\cdot\|$. This is reflected in (1.1), but by the principle of the estimate the number 1 is between the lower and upper bound. Although being individually sharp, the upper and lower bound differ by at least $\mathrm{tr}(E)$. That is also true for the other bounds mentioned.

Notable exceptions are papers by Ostrowski [14] and Hans Schneider [16], proving bounds depending on the trace and on the absolute row sums of $E$. If all elements of $E$ are bounded by $\varepsilon$ in absolute value, then either difference between upper and lower bound is $\mathcal{O}(n^3\varepsilon^2)$.

We improve that into

$$\exp\left(\mathrm{tr}(E) - \frac{\epsilon^2/2}{1-\epsilon}\right) \le \det(I+E) \le \exp\left(\mathrm{tr}(E)+\frac{\epsilon^2}{2}\right), \tag{1.3}$$

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provided that the Frobenius norm $\|E\|_F =: \epsilon$ is less than 1, so that the bounds are $\exp\left(\mathrm{tr}(E) \pm \dfrac{\epsilon^2}{2}\right)$ up to $\mathcal{O}(\varepsilon^3)$. For complex $E$, the bounds apply to $|\det(I+E)|$. We need the following lemma.

LEMMA 1.1. *Let $E$ be a real or complex $n \times n$ matrix. Then*

$$|\det(I+E)| = \exp\left(\frac{1}{2}\sum_{k=1}^{n} \log(1 + 2\,\Re(\lambda_k) + |\lambda_k|^2)\right) \tag{1.4}$$

*with the conventions* $\log 0 := -\infty$ *and* $\exp(-\infty) := 0$.

PROOF. Using $\det(I+E) = \prod_{k=1}^{n}(1+\lambda_k) = \exp\left(\sum_{k=1}^{n}\log(1+\lambda_k)\right)$ we obtain

$$|\det(I+E)| = \left|\exp\left(\sum_{k=1}^{n}\log(1+\lambda_k)\right)\right| = \exp\left(\sum_{k=1}^{n}\Re(\log(1+\lambda_k))\right) = \exp\left(\sum_{k=1}^{n}\log(|1+\lambda_k|)\right),$$

and the result follows. □

THEOREM 1.2. *Let $E$ be a real or complex $n \times n$ matrix. Suppose its eigenvalues $\lambda_k$ satisfy $\Re(\lambda_k) > -1$, and abbreviate its Frobenius norm by $\epsilon := \|E\|_F$. Then*

$$\mathrm{tr}(E) - \frac{\epsilon^2/2}{1 + \min\limits_j \Re(\lambda_j)} \le \log|\det(I+E)| \le \mathrm{tr}(E) + \frac{\epsilon^2}{2}. \tag{1.5}$$

*Denote the spectral radius by $\rho(\cdot)$. If $\rho(|E|) < 1$, then*

$$\mathrm{tr}(E) - \frac{\epsilon^2/2}{1 - \rho(|E|)} \le \log|\det(I+E)|. \tag{1.6}$$

*If $\epsilon < 1$, then*

$$\mathrm{tr}(E) - \frac{\epsilon^2/2}{1-\epsilon} \le \log|\det(I+E)| \le \mathrm{tr}(E) + \frac{\epsilon^2}{2}. \tag{1.7}$$

*For real $E$,* $\det(I+E) = |\det(I+E)|$.

PROOF. We use [8] $\sum_{k=1}^{n}|\lambda_k^2| \le \sum_{k=1}^{n}\sigma_k^2 = \epsilon^2$, where $\sigma_k$ denote the singular values of $E$, and [10, 4.5.1] $\log(1+\alpha) \le \alpha$ for $-1 < \alpha \in \mathbb{R}$. Furthermore[1],

$$-\log(1-\beta) = \beta + \sum_{i=2}^{\infty}\frac{\beta^i}{i} \le \beta + \frac{\beta^2/2}{1-\beta} \qquad \text{for } \beta \in (0,1],$$

so that

$$\alpha - \frac{\alpha^2/2}{1-\alpha} \le \log(1+\alpha) \le \alpha \qquad \text{for } -1 < \alpha \in \mathbb{R}. \tag{1.8}$$

Applying Lemma 1.1, the upper bound follows by

$$\begin{aligned} \log|\det(I+E)| &\le \tfrac{1}{2}\sum_{k=1}^{n} 2\,\Re(\lambda_k) + |\lambda_k|^2 \\ &= \mathrm{tr}(E) + \tfrac{1}{2}\sum_{k=1}^{n}|\lambda_k|^2 \\ &\le \mathrm{tr}E + \tfrac{1}{2}\epsilon^2. \end{aligned}$$

[1] Thanks to Richard Brent for pointing to the factor 1/2.

For the lower bound, $1+2\,\Re(\lambda_k)+|\lambda_k|^2 \ge (1+\Re(\lambda_k))^2$, Lemma 1.1 and (1.8) imply

$$\begin{aligned}
\log|\det(I+E)| &\ge \sum_{k=1}^{n} \log(1+\Re(\lambda_k)) \\
&\ge \operatorname{tr}(E) - \sum_{k=1}^{n} \frac{(\Re(\lambda_k))^2/2}{1+\Re(\lambda_k)} \\
&\ge \operatorname{tr}(E) - \tfrac{1}{2}\big(1+\min_j \Re(\lambda_j)\big)^{-1} \sum_{k=1}^{n} (\Re(\lambda_k))^2 \\
&\ge \operatorname{tr}(E) - \frac{\epsilon^2/2}{1+\min_j \Re(\lambda_j)}.
\end{aligned}$$

The lower bounds in (1.6) and (1.7) follow by Perron-Frobenius Theory and $-\Re(\lambda_j) \le \rho(E) \le \rho(|E|) \le \epsilon$. For real $E$, $\det(I+E) = \prod_{k=1}^{n}(1+\lambda_k)$ and $\Re(\lambda_k) > -1$ imply $\det(I+E) > 0$. □

In [2, 3] the authors show that either bound in (1.1) is sharp. For the lower bound they consider the symmetric Toeplitz matrix $T$ with first row $(\delta, \varepsilon, \ldots, \varepsilon)$ and show that its eigenvalues are $\delta + (n-1)\varepsilon$ with multiplicity 1, and $\delta - \varepsilon$ with multiplicity $n-1$. Thus

$$\det(I-T) = (1-\delta-(n-1)\varepsilon)(1-\delta+\varepsilon)^{n-1} \tag{1.9}$$

is equal to the lower bound in (1.1). Since $T$ is non-negative with identical row sums, $\rho(T) = \rho(|T|) = \delta + (n-1)\varepsilon$, and the lower bound in (1.6) reads

$$\log\det(I-T) \ge -n\delta - \frac{(n\delta^2 + n(n-1)\varepsilon^2)/2}{1-\sqrt{n\delta^2 + n(n-1)\varepsilon^2}}.$$

This is of order $\mathcal{O}(\delta^2+\varepsilon^2)$ weaker than the lower bound in (1.1). The same matrix can be used to see that the lower bound in (1.1) can be weak:

$$\det(I+T) = (1+\delta+(n-1)\varepsilon)(1+\delta-\varepsilon)^{n-1}$$

is $1+n\delta+\mathcal{O}(\delta^2+\varepsilon^2)$, but the lower bound in (1.1) is still (1.9), thus $1-n\delta+\mathcal{O}(\delta^2+\varepsilon^2)$. In contrast, (1.6) yields $\det(I+T) \ge 1+n\delta+\mathcal{O}(\delta^2+\varepsilon^2)$.

For the upper bound in (1.1), a skew-symmetric Hadamard matrix $H = I+G$ of order $n$, whenever it exists, is used in [2, 3], which means $H^TH = nI$ and $G^T = -G$. For $E := \varepsilon G$ it follows $(I+\varepsilon G)^T(I+\varepsilon G) = I + (n-1)\varepsilon^2 I$ and $\det(I+\varepsilon G) = (1+(n-1)\varepsilon^2)^{n/2}$. Since the maximal diagonal element of $E$ is zero, this is the upper bound in (1.1). Our bound (1.5) computes to

$$\log\det(I+\varepsilon G) \le \frac{n(n-1)}{2}\varepsilon^2$$

which is weaker by $\mathcal{O}(\varepsilon^4)$. For an example where the upper bound in (1.1) is weak, consider $E = -\varepsilon H = -\varepsilon I - \varepsilon G$. Then $(I-\varepsilon H)^T(I-\varepsilon H) = (1-\varepsilon)^2 I + (n-1)\varepsilon^2 I$ shows $\det(I+E) = (1-2\varepsilon+n\varepsilon^2)^{n/2}$. For the upper bound in (1.1) use $\delta = \varepsilon$ and obtain $\det(I+E) \le (1+2\varepsilon+n\varepsilon^2)^{n/2} = 1+n\epsilon+\mathcal{O}(\varepsilon^2)$, whereas (1.5) reads

$$\log\det(I+E) \le -n\varepsilon + \frac{n^2}{2}\varepsilon^2,$$

and this is sharp up to $\mathcal{O}(\varepsilon^2)$.

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