Almost sharp bounds on the componentwise distance to the nearest singular matrix*

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Abstract

The normwise distance of a regular matrix $A \in M_n(\mathbb{R})$ to the nearest singular matrix is well known to be $\|A^{-1}\|^{-1}$. Such a normwise distance neglects small entries in the matrix, and it does not allow for weights in a perturbation. The reciprocal $\|\,|A^{-1}| \cdot E\|^{-1}$ of the Bauer-Skeel condition number is known to be a *lower* bound for the *componentwise* distance of A to the nearest singular matrix weighted by the nonnegative matrix E. In this paper we derive an *upper* bound for this componentwise distance involving the Bauer-Skeel condition number. We show that this upper bound is sharp up to a constant factor less than $3 + 2\sqrt{2}$, independent of A and E. For finite values of n, improved constants are given as well.

0 Introduction

It is well-known that the normwise distance of a regular matrix $A \in M_n(\mathbb{R})$ to the nearest singular matrix is equal to $\|A^{-1}\|^{-1}$. Such a normwise distance neglects small entries in the matrix, and it neglects possible weights for a perturbation. For example, for the matrix (cf. [3]) $A = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 2\varepsilon & 2\varepsilon \\ 1 & 2\varepsilon & -\varepsilon \end{pmatrix}$ there exists a matrix Δ with $\|\Delta\|_2 = 2 \cdot\, \varepsilon$ and $A + \Delta$ singular. On the other hand, any *relative perturbation* less than 0.37 of the individual components of the matrix A *cannot* produce a singular matrix. This leads to the definition of the componentwise distance $\sigma(A, E)$ to the nearest singular matrix weighted by some nonnegative $E \in M_n(\mathbb{R})$:

$$\sigma(A, E) := \min\{\, \alpha \in \mathbb{R} \mid |A'_{ij} - A_{ij}| \le \alpha \cdot E_{ij} \text{ for some singular } A'\}. \tag{1}$$

*Linear and Multilinear Algebra (LAMA), 42:93–107, 1997

If no such α exists, we define $\sigma(A,E) := \infty$.

Poljak and Rohn (cf. [7]) showed that the computation of the componentwise distance to the nearest singular matrix $\sigma(A,E)$ is NP-hard.

A condition number taking weights E_{ij} into account is the Bauer-Skeel condition number $\text{cond}_{BS}(A,E) := \| \, |A^{-1}| \cdot E \|$. In our example, $\text{cond}_{BS}(A,|A|) = 2$ for the 2-norm. Improper scaling of the matrix may lead to a large value of the Bauer-Skeel condition number. It has been shown by Demmel [1] that the *minimum* Bauer-Skeel condition number achievable by diagonal scaling can be explicitly calculated for p-norms, namely (ρ denotes the spectral radius)

$$\min_{D_1,D_2} \ \text{cond}_{BS}(D_1AD_2, D_1ED_2) = \rho(\, |A^{-1}| \cdot E). \tag{2}$$

D_1, D_2 are regular diagonal matrices. D_1 can be omitted because the Bauer-Skeel condition number is invariant under row scaling. On the other hand, the inverse of this number is a well-known and easy-to-prove lower bound for the componentwise distance to the nearest singular matrix weighted by E:

$$\frac{1}{\rho(\, |A^{-1}| \cdot E)} \le \sigma(A,E).$$

One may ask, whether - like for normwise distances - a large (minimum) Bauer-Skeel condition number implies that not too far away in *a componentwise sense* there exists a singular matrix. More precisely, one may ask whether there are finite constants $\gamma(n) \in \mathrm{I\!R}$ such that for any regular $A \in M_n(\mathrm{I\!R})$ and nonnegative $E \in M_n(\mathrm{I\!R})$ there holds

$$\frac{1}{\rho(\, |A^{-1}| \cdot E)} \le \sigma(A,E) \le \frac{\gamma(n)}{\rho(\, |A^{-1}| \cdot E)}. \tag{3}$$

For $E = |A|$ this has been conjectured by N.J. Higham and J. Demmel [1], see also [4]. For general E it is proved in [8] that $\gamma(n) \le 2.321 \cdot n^{1.7}$ with an asymptotic upper bound n^{1+ln2}. Moreover, it has been shown over there that validity of (3) implies $\gamma(n) \ge n$. In the present note we use results obtained in [9] to prove the following bound for $\gamma(n)$, which is sharp up to a small constant factor:

$$n \le \gamma(n) \le (3 + 2\sqrt{2}) \cdot n. \tag{4}$$

For smaller values of n better bounds will be given.

1 Notation and basic results

We use standard notation from matrix theory, cf. [5], [6]. In particular, Q_{kn} denotes the set of k-tuples of strictly increasing integers out of $\{1, \ldots, n\}$. For $\omega \in Q_{kn}$, $A[\omega] \in M_k(\mathbb{R})$ denotes the principal submatrix of A consisting of rows and columns in ω. We denote the identity matrix of proper dimension by I, and by $(\mathbf{1}) \in \mathbb{R}^n$ the vector with all components equal to 1.

Definition 1.1. A set $\omega = \{\omega_1, \ldots, \omega_k\}$ of mutually different integers ω_i out of $\{1, \ldots, n\}$ defines a *cycle*

$$A_\omega := \{A_{\omega_1\omega_2}, \ldots, A_{\omega_{k-1}\omega_k}, A_{\omega_k\omega_1}\}.$$

The *length* k of a cycle is denoted by $|\omega| := k$. Any cycle defines a *cyclic product*

$$\prod A_\omega := \prod_{i=1}^{|\omega|} A_{\omega_i\omega_{i+1}} \text{ with } \omega_{|\omega|+1} := \omega_1.$$

Note that any diagonal element forms a cycle of length 1. Diagonal similarity transformations do not change the value of any cyclic product. It is well-known that for any nonzero cyclic product there exists a diagonal matrix D such that all elements in $(D^{-1}A\,D)_\omega$ are equal in absolute value (see for example [8]), namely equal to the geometric mean $|\prod A_\omega|^{1/|\omega|}$ of the elements of $|A_\omega|$:

$$\prod A_\omega \neq 0 \quad \Rightarrow \quad \exists \text{ diagonal } D : |D^{-1}A\,D|_{\omega_i\omega_{i+1}} = |\prod A_\omega|^{1/|\omega|} \text{ for } 1 \leq i \leq |\omega|. \tag{5}$$

Throughout the paper, we use comparison and absolute value of vectors and matrices *entrywise* (for a cycle ω, $|\omega|$ denotes the length). For example, $E \geq 0$ for $E \in M_n(\mathbb{R})$ means $E_{ij} \geq 0$ for all i, j, and a short notation for (1) is

$$\sigma(A, E) := \min\{\, \alpha \in \mathbb{R} \mid |A' - A| \leq \alpha \cdot E \text{ for some singular } A' \,\}.$$

The set of signature matrices $\mathcal{S}$ consists of diagonal matrices S with diagonal entries in $\{-1, +1\}$, i.e. $S \in \mathcal{S} \Leftrightarrow |S| = I$. The *real spectral radius* $\rho_0(A)$ of $A \in M_n(\mathbb{R})$ is defined by

$$\rho_0(A) = \max\{\, |\lambda| \mid \lambda \;\; \textit{real} \text{ eigenvalue of } A \,\}. \tag{6}$$

If A has no real eigenvalues, we define $\rho_0(A) := 0$.

For singular $A \in M_n(\mathbb{IR})$, we have $\sigma(A,E) = 0$ for any $0 \le E \in M_n(\mathbb{IR})$. Assume that A is regular. Then for $\widetilde{E} \in M_n(\mathbb{IR})$, $|\widetilde{E}| \le E$ there holds

$$A - \widetilde{E} = A \cdot (I - A^{-1}\widetilde{E}),$$

which means that singularity of $A - \widetilde{E}$ is equivalent to the fact that $A^{-1}\widetilde{E}$ has the real eigenvalue 1. This implies

$$\sigma(A,E) = \left\{ \max_{|\widetilde{E}| \le E} \rho_0(A^{-1}\widetilde{E}) \right\}^{-1}. \tag{7}$$

In [9], the *sign-real spectral radius* $\rho_0^S(A)$ has been defined and investigated:

$$\rho_0^S(A) := \max_{S \in \mathcal{S}} \ \rho_0(S \cdot A).$$

Many interesting properties and Perron-Frobenius like theorems have been proved over there, among them

$$\rho_0^S(A) \text{ depends continuously on the entries of } A, \tag{8}$$

$$\rho_0^S(A) = \rho_0^S(D^{-1}\, A\, D) = \rho_0^S(S_1\, A\, S_2) \text{ for regular diagonal } D \text{ and } S_1, S_2 \in \mathcal{S}\ , \tag{9}$$

$$\rho_0^S(A) \ge \max_i \ |A_{ii}|, \quad \text{and} \quad \rho_0^S(A) \ge \rho_0^S(A[\omega]) \text{ for } \omega \in Q_{kn}, \tag{10}$$

$$\rho_0^S(A) = \max_{x \in \mathbb{IR}^n} \ \min_{x_i \ne 0} \ \left| \frac{(Ax)_i}{x_i} \right|, \tag{11}$$

$$\sigma(A,E) = \left[\rho_0^S \begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix} \right]^{-2}. \tag{12}$$

By (12) it follows that computation of the sign-real spectral radius ρ_0^S is also NP-hard. We will use the sign-real spectral radius to prove (4).

Combining (7) and (11) yields

$$\sigma(A,E) = \frac{1}{\max\limits_{|\widetilde{E}| \le E} \rho_0(A^{-1}\widetilde{E})} = \frac{1}{\max\limits_{|\widetilde{E}| \le E} \rho_0^S(A^{-1}\widetilde{E})} = \frac{1}{\max\limits_{|\widetilde{E}| \le E} \ \max\limits_{x \in \mathbb{IR}^n} \ \min\limits_{x_i \ne 0} \left| \dfrac{(A^{-1}\widetilde{E} \cdot x)_i}{x_i} \right|}. \tag{13}$$

Henceforth, any $\widetilde{E}$ with $|\widetilde{E}| \le E$ and any $0 \ne x \in \mathbb{IR}^n$ yield an upper bound for $\sigma(A,E)$. We will construct a proper matrix $\widetilde{E} \in M_n(\mathbb{IR})$ with $|\widetilde{E}| \le E$, and choose some appropriate $x \in \mathbb{IR}^n$ in order to obtain a suitable upper bound of $\sigma(A,E)$. This is the key to our proof of the announced new and almost sharp bound (4) on $\gamma(n)$.

2 Main results

The first step in finding an upper bound on $\sigma(A, E)$ using (13) is the following lower bound on $\rho_0^S(A)$. It is expressed by the geometric mean of the elements of a cycle of A.

Lemma 2.1. For $A \in M_n(\mathrm{I\!R})$ and any cycle ω there holds

$$\rho_0^S(A) \geq |\prod A_\omega|^{1/|\omega|} \cdot (3 + 2\sqrt{2})^{-1}.$$

Proof by induction. For $|\omega| = 1$ the lemma follows by (10). Assume $|\omega| > 1$ and $\prod A_\omega \neq 0$. Suitable renumbering puts ω into $(1, \ldots, |\omega|)$, and the inheritance property (10) of $\rho_0^S(A)$ allows us to assume w.l.o.g. $\omega = (1, \ldots, n)$. By (9), the sign-real spectral radius is invariant under diagonal similarity transformations, and by (5) we may assume all elements in A_ω to be equal in absolute value. Proper scaling and observing $\rho_0^S(cA) = |c| \cdot \rho_0^S(A) = |c| \cdot \rho_0^S(S \cdot A)$ for any $S \in \mathcal{S}$, $c \in \mathrm{I\!R}$ shows that we may assume w.l.o.g.

$$\omega = (1, \ldots, n), \quad \text{and all elements in } A_\omega \text{ are equal to } 1 \ .$$

Moreover, in view of the assertion, we may suppose ΠA_ω to be a cyclic product of maximum absolute value. This implies $|A_{ij}| \leq 1$ for all i, j, because any A_{ij} forms a cycle together with suitable elements in the full cycle A_ω. Summarizing, we have shown that we may assume w.l.o.g.

$$\begin{array}{l} \omega = \{1, \ldots, n\} \ , \\ A_{12} = A_{23} = \ldots = A_{n-1,n} = A_{n1} = 1, \quad \text{and} \quad |A_{ij}| \leq 1 \text{ for } 1 \leq i, j \leq n \ . \end{array} \tag{14}$$

We split A into

$$A = \begin{pmatrix} \diagdown & & 0 \\ & * & \diagdown \\ 0 & \cdots & 0 \quad 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & 0 \\ 0 & & 0 & 1 \\ 1 & & & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \diagdown & * \\ & \ddots & \ddots & \diagdown \\ 0 & & 0 & 0 \\ 0 & * & \cdots & * \end{pmatrix}$$

$$= \quad \mathrm{L} \quad + \quad \mathrm{P} \quad + \quad \mathrm{U} \ . \tag{15}$$

More precisely,

$$L_{ij} := \begin{cases} A_{ij} & \text{for } i \geq j \quad \text{and } i \neq n \\ 0 & \text{otherwise} \end{cases} ,$$

P is the cyclic shift with $p = 1$ for $p \in P_\omega$, and $U := A - L - P$. Next, we show that for any nonnegative vector $x \in \mathbb{R}^n$ there are signature matrices $S, T \in \mathcal{S}$ with

$$(S \cdot L \cdot T) \cdot x \geq 0 \quad \text{and} \quad (S \cdot P \cdot T) \cdot x \geq 0 \; . \tag{16}$$

For this purpose, we first construct T recursively such that for $1 \leq i \leq n-1$

$$(L \cdot T \cdot x)_i \cdot (P \cdot T \cdot x)_i \geq 0 \; . \tag{17}$$

This is achieved by the following algorithm:

$T := I$;

for $i := 1$ to $n-1$ do

if $(L \cdot T \cdot x)_i < 0$ then

for $\nu := 1$ to i do $T_{\nu\nu} := -T_{\nu\nu}$;

Note that $(P \cdot T \cdot x)_i = T_{i+1,i+1} \cdot x_{i+1}$, and that the case $i = n$ is excluded in (17). In the for-loop, for the current value of i it is $(PTx)_i \geq 0$ by definition, and execution of the if-statement assures (17) for the current value of i. But (17) remains also valid for the previous indices because *all* signs of the $T_{\nu\nu}$, $1 \leq \nu \leq i$ are inverted. Hence (17) is valid for $1 \leq i \leq n-1$.

Next, we define $S \in \mathcal{S}$ by $S_{ii} := \mathrm{sign}(\,(P \cdot T)_{i,i+1}) = T_{i+1,i+1}$ for $1 \leq i \leq n-1$, and we set $S_{nn} := 1$. This yields the right inequality in (16), and with (17) also the left inequality of (16).

Now we define

$$q := 1 - \sqrt{2}/2 \quad \text{and} \quad x := (q, q^2, \ldots, q^n)^T \in \mathbb{R}^n. \tag{18}$$

By (9), the sign-real spectral radius of A is invariant under multiplication of A by signature matrices from left or right. Using this together with (11) and the q and x as defined in (18) yields

$$\rho_0^S(A) = \rho_0^S(S\,A\,T) \;\geq\; \min_i \left| \frac{(S\,A\,T\,x)_i}{x_i} \right| \;=\; \min_i \; q^{-i} \cdot |S \cdot (L + P + U) \cdot T \cdot x|_i \; .$$

From (14) we know $|S \cdot U \cdot T|_{ij} \leq 1$. Moreover, $(S\,P\,T)x \geq 0$ from (16), and $x > 0$ implies $S\,P\,T\,x = Px$. Hence, in view of (16), there holds for $1 \leq i \leq n-1$

$$\begin{aligned}
\left|\frac{(S\,A\,T\,x)_i}{x_i}\right| &= q^{-i} \cdot [|S \cdot (L+P+U) \cdot T \cdot x|]_i \\
&\geq q^{-i} \cdot [\,(S \cdot L \cdot T + S \cdot P \cdot T - |U|) \cdot x\,]_i \\
&\geq q^{-i} \cdot [\,(P - |U|) \cdot x]_i \\
&\geq q^{-i} \cdot (q^{i+1} - \sum_{\nu=i+2}^{n} q^{\nu}) \geq q \cdot (2 - \frac{1}{1-q}) = (3 + 2 \cdot \sqrt{2})^{-1},
\end{aligned}$$

and similarly for $i = n$,

$$\begin{aligned}
\left|\frac{(S\,A\,T\,x)_n}{x_n}\right| &= q^{-n} \cdot [|S \cdot (L+P+U) \cdot T \cdot x|]_n \geq q^{-n} \cdot (q - \sum_{\nu=2}^{n} q^{\nu}) \\
&\geq q^{1-n} \cdot (2 - \frac{1}{1-q}) > (3 + 2 \cdot \sqrt{2})^{-1}.
\end{aligned}$$

The max min characterization (11) of $\rho_0^S(A)$ proves the lemma. ■

For small values of $|\omega|$ the bound in Lemma 2.1 can be improved. For example, for $|\omega| = 1$ or $|\omega| = 2$ the constant $(3 + 2\sqrt{2})^{-1}$ in Lemma 2.1 can be replaced by 1, i.e.

$$\rho_0^S(A) \geq |\prod A_\omega|^{1/|\omega|} \qquad \text{for} \quad |\omega| = 1 \text{ or } |\omega| = 2,$$

(cf. (10) and [8], Theorem 6.5). This is no longer true for $|\omega| \geq 3$, as is seen by

$$A = \begin{pmatrix} -0.3 & 1 & -0.8 \\ -0.8 & -0.3 & 1 \\ 1 & -0.8 & -0.3 \end{pmatrix} \quad \text{with } \rho_0^S(A) < 0.95.$$

However, we can improve the constant $(3 + 2\sqrt{2})$ in Lemma 2.1 in the following way. We proceed by induction over n and assume

$$\rho_0^S(A) \geq |\prod A_\alpha|^{1/|\alpha|} \cdot \psi_{|\alpha|}^{-1} \quad \text{for all} \quad |\alpha| < n, \tag{19}$$

where $\psi_1 = \psi_2 = 1$. We will construct a ψ_n satisfying (19) in several steps. First, we show that w.l.o.g. we may assume $|A|$ to be bounded by a circulant, second, the problem is reduced to an eigenvalue problem and finally, we show that ψ_n is the unique positive value solving this eigenvalue problem. A posteriori, this is the definition of ψ_n satisfying (19).

Using the same arguments as in the proof of Lemma 2.1 together with proper scaling we may assume w.l.o.g.

$$\omega = \{1, \ldots, n\}, \quad A_{12} = A_{13} = \ldots = A_{n-1,n} = A_{n1} = 1. \tag{20}$$

Hence, we want to find ψ_n such that $\rho_0^S(A) \geq \psi_n^{-1}$ for a matrix satisfying (20).

Suppose $|\prod A_\alpha|^{1/|\alpha|} \geq \psi_{|\alpha|} \cdot \psi_n^{-1}$ for some cycle α with $1 \leq |\alpha| \leq n-1$. Then (19) implies $\rho_0^S(A) \geq |\prod A_\alpha|^{1/|\alpha|} \cdot \psi_{|\alpha|}^{-1} \geq \psi_{|\alpha|} \cdot \psi_n^{-1} \cdot \psi_{|\alpha|}^{-1} = \psi_n^{-1}$. Therefore, we may assume w.l.o.g. $|\prod A_\alpha|^{1/|\alpha|} < \psi_{|\alpha|} \cdot \psi_n^{-1}$ for all cycles α with $1 \leq |\alpha| \leq n-1$.

For $\alpha = \{k\}$, $1 \leq k \leq n$ this means $|\prod A_\alpha|^{1/|\alpha|} = |A_{kk}| < \psi_n^{-1}$. For $\alpha = \{1, 2\}$ this implies

$$|\prod A_\alpha|^{1/|\alpha|} = |A_{12}\, A_{21}|^{1/2} = |A_{21}|^{1/2} < \psi_2/\psi_n, \text{ and therefore } |A_{21}| < (\psi_2/\psi_n)^2.$$

Setting $\alpha = \{2, 3\}, \ldots, \{n, 1\}$ this implies $|A_{32}| < (\psi_2/\psi_n)^2, \ldots, |A_{1n}| \leq (\psi_2/\psi_n)^2$. For $\alpha = \{1, 2, 3\}$ we have

$$|\prod A_\alpha|^{1/|\alpha|} = |A_{12}\, A_{23}\, A_{31}|^{1/3} = |A_{31}|^{1/3} < \psi_3/\psi_n, \text{ and therefore } |A_{31}| < (\psi_3/\psi_n)^3.$$

Proceeding in this way for $\alpha = \{2, 3, 4\}, \ldots$ and so forth, we may assume w.l.o.g. that $|A|$ is bounded by the following circulant

$$|A| \leq \begin{pmatrix} c_1 & 1 & c_{n-1} & \cdots & & c_3 & c_2 \\ c_2 & c_1 & 1 & c_{n-1} & \cdots & & c_3 \\ & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & c_1 & 1 \\ 1 & c_{n-1} & & \cdots & & c_2 & c_1 \end{pmatrix} =: C \quad \text{with } c_i := (\psi_i/\psi_n)^i \ .$$

Outside the cycle $\omega = \{1, \ldots, n\}$ we may even assume strong inequality. Note that ψ_i, $1 \leq i \leq n-1$ is already known by induction hypothesis, but ψ_n is not. That means, the upper bound of $|A|$ depends on ψ_n, the quantity we are looking for, and we wish to prove $\rho_0^S(A) \geq \psi_n^{-1}$. Let $C = \widetilde{L} + P + \widetilde{U}$ be a splitting like in (15). Then $\widetilde{U}$ depends on ψ_n. Suppose, $P - \widetilde{U}$ has a positive eigenvector x with positive eigenvalue ψ_n^{-1}, i.e. $(P - \widetilde{U})x = \psi_n^{-1} \cdot x > 0$. Then we may proceed as in the proof of Lemma 2.1, split $A = L + P + U$ as in (15), and assume w.l.o.g. $L \cdot x \geq 0$ and $P \cdot x \geq 0$. It is $|U| \leq \widetilde{U}$, and for $1 \leq i \leq n$,

$$\left| \frac{(Ax)_i}{x_i} \right| \geq x_i^{-1} \cdot [\, (L + P - |U|) \cdot x]_i \geq x_i^{-1} \cdot [\, (P - \widetilde{U}) \cdot x]_i = \psi_n^{-1} \ ,$$

and the max min characterization (11) yields $\rho_0^S(A) \geq \psi_n^{-1}$.

The problem remains to find ψ_n such that ψ_n^{-1} is a positive eigenvalue to a positive eigenvector of $P - \widetilde{U}$ as defined above. For $n = 3$ this means

$$\begin{pmatrix} 0 & 1 & -(\psi_2/\psi_3)^2 \\ 0 & 0 & 1 \\ 1 & -(\psi_2/\psi_3)^2 & -\psi_1/\psi_3 \end{pmatrix} \cdot x = \psi_3^{-1} \cdot x.$$

Following an idea by Ludwig Elsner [2] one can prove that the ψ_n exist and are uniquely defined. Set $\widetilde{U}_{\psi_n} = \widetilde{U}$ to indicate the dependency of $\widetilde{U}$ on ψ_n. Then $P - \widetilde{U}_{\psi_n} = P \cdot (I - P^T \cdot U_{\psi_n}) = P \cdot M(\psi_n)$ with

$$M(\psi_n) := \begin{pmatrix} 1 & -c_{n-1} & -c_{n-2} & \cdots & -c_1 \\ & 1 & -c_{n-1} & \cdots & -c_2 \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}, \quad c_i := (\psi_i/\psi_n)^i \,. \tag{21}$$

For any $\psi_n > 0$, $M(\psi_n)^{-1}$ exists and is nonnegative upper triangular. Therefore, $(P - \widetilde{U}_{\psi_n})^{-1} = M(\psi_n)^{-1} \cdot P^T$ is nonnegative irreducible, and it has a uniquely determined positive eigenvalue $\rho\Big((P - \widetilde{U}_{\psi_n})^{-1}\Big)$ with positive eigenvector. Furthermore, the Neumann series for $(I - P^T \cdot U_{\psi_n})^{-1}$ shows that the Perron root of $(P - \widetilde{U}_{\psi_n})^{-1}$ is strictly decreasing with increasing ψ_n. Henceforth, there must be a unique value for ψ_n such that $\rho\Big((P - \widetilde{U}_{\psi_n})^{-1}\Big) = \rho(M(\psi_n)^{-1} \cdot P^T) = \psi_n$, and ψ_n^{-1} is a positive eigenvalue of $P - \widetilde{U}_{\psi_n}$ to a positive eigenvector.

When calculating the ψ_n by using $(P - \widetilde{U})^{-1}$ explicitly or implicitly, the numerical computation becomes instable. We used instead the Neumann series for $(P - \widetilde{U})^{-1}$ and obtained the following results for $1 \leq n \leq 36$.

$\psi_1 = 1.0000$	$\psi_{10} = 3.3745$	$\psi_{19} = 4.2618$	$\psi_{28} = 4.6803$
$\psi_2 = 1.0000$	$\psi_{11} = 3.5187$	$\psi_{20} = 4.3227$	$\psi_{29} = 4.7134$
$\psi_3 = 1.5874$	$\psi_{12} = 3.6472$	$\psi_{21} = 4.3790$	$\psi_{30} = 4.7447$
$\psi_4 = 1.9656$	$\psi_{13} = 3.7625$	$\psi_{22} = 4.4313$	$\psi_{31} = 4.7743$
$\psi_5 = 2.2920$	$\psi_{14} = 3.8664$	$\psi_{23} = 4.4800$	$\psi_{32} = 4.8023$
$\psi_6 = 2.5731$	$\psi_{15} = 3.9605$	$\psi_{24} = 4.5254$	$\psi_{33} = 4.8289$
$\psi_7 = 2.8161$	$\psi_{16} = 4.0460$	$\psi_{25} = 4.5679$	$\psi_{34} = 4.8541$
$\psi_8 = 3.0272$	$\psi_{17} = 4.1242$	$\psi_{26} = 4.6077$	$\psi_{35} = 4.8781$
$\psi_9 = 3.2119$	$\psi_{18} = 4.1959$	$\psi_{27} = 4.6451$	$\psi_{36} = 4.9010$

Table 2.2 Values for ψ_n

A graph for larger values of ψ_n looks as follows.

Graph 2.3. Graph of ψ_n

We do not know whether the bound in Lemma 2.1 can be achieved asymptotically. The proof of Lemma 2.1 can be regarded as finding the positive eigenvalue λ of $(P - \widetilde{U})^{-1}$ with $P^T \widetilde{U}$ being strictly upper triangular with all components equal to 1 above the diagonal. This implies $\psi_n < (3 + 2\sqrt{2})$ for all n. It has been shown by L. Elsner and S. Friedland [2] that λ converges to $(3+2\sqrt{2})^{-1}$ for $n \to \infty$. Graph 2.3 shows that for larger n the values of ψ_n are not too far from $(3 + 2\sqrt{2})$. In fact, for $n = 500$ the difference is less than 0.08. We do not know the limit of the ψ_n for $n \to \infty$.

Summarizing, we have the following result.

Theorem 2.4. For $A \in M_n(\mathrm{I\!R})$ and any cycle ω there holds

$$\rho_0^S(A) \geq |\prod A_\omega|^{1/|\omega|} \cdot \psi_{|\omega|}^{-1} \geq |\prod A_\omega|^{1/|\omega|} \cdot (3 + 2\sqrt{2})^{-1},$$

where $\psi_1 := \psi_2 := 1$, and ψ_k, $k > 2$ is defined recursively to be the unique positive number such that ψ_k is the Perron root of $M(\psi_k)^{-1} \cdot P^T$, where $M(\psi_k)$ is defined in (21).

Some values of ψ_n are listed in Table 2.2 and shown in Graph 2.3. Theorem 2.4 may be useful for practical applications because it frequently gives a reasonable and simple to compute lower bound on $\rho_0^S(A)$. For short cycles, the constant $\psi_{|\omega|}$ is especially favourable.

We need the following technical lemma to prove our main result.

Lemma 2.5. Let regular $A \in M_n(\mathrm{I\!R})$ and $0 \le E \in M_n(\mathrm{I\!R})$ be given, and suppose $|A^{-1}| \cdot E$ is row stochastic. Then

$$\sigma(A, E) \le n \cdot \psi_n,$$

where ψ_n is defined as in Theorem 2.4.

Proof. We will construct a matrix $\widetilde{E} \in M_n(\mathrm{I\!R})$, $|\widetilde{E}| \le E$ with $\rho_0^S(A^{-1}\widetilde{E}) \ge \{n \cdot \psi_n\}^{-1}$. Define $C := |A^{-1}| \cdot E$ and let C_{i,m_i} be maximum row elements, i.e.

$$C_{i,m_i} = \max_{\nu} C_{i\nu}.$$

C is row stochastic, and therefore $C_{i,m_i} \ge n^{-1}$ for $1 \le i \le n$. Within the elements $\{ C_{i,m_i} \mid 1 \le i \le n \}$ there must be a cycle of length k, $1 \le k \le n$, and suitable renumbering puts this cycle into $\{1, \ldots, k\}$. Hence, we may assume w.l.o.g.

$$c \ge n^{-1} \text{ for all } c \in C_\omega \text{ and } \omega = \{1, \ldots, k\}, 1 \le k \le n.$$

Define $\widetilde{E} \in M_n(\mathrm{I\!R})$ by

$$\widetilde{E}_{ij} := \begin{cases} \operatorname{sign}(\,(A^{-1})_{j-1,i}) \cdot E_{ij} & \text{for} \quad 1 \le i \le n, \quad 2 \le j \le k \\ \operatorname{sign}(\,(A^{-1})_{ki}) \cdot E_{ij} & \text{for} \quad 1 \le i \le n, \quad j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then $|\widetilde{E}| \le E$, and for $\widetilde{C} := A^{-1}\widetilde{E}$ there holds

$$\widetilde{C}_{i,i+1} = \sum_{\nu=1}^{n} (A^{-1})_{i\nu} \cdot \operatorname{sign}(\,(A^{-1})_{i\nu}) \cdot E_{\nu,i+1} = C_{i,i+1} \ge n^{-1},$$

and similarly $\widetilde{C}_{k1} = C_{k1} \ge n^{-1}$. Hence, $|\prod \widetilde{C}_\omega|^{1/|\omega|} \ge n^{-1}$, and (13) and Theorem 2.4 yield $\sigma(A, E) \le \rho_0^S(A^{-1}\widetilde{E})^{-1} \le n \cdot \psi_k \le n \cdot \psi_n$. ■

With these preliminaries we can prove the following result: if the minimum Bauer-Skeel condition number achievable by column scaling is still large, then a singular matrix cannot be too far away in the componentwise sense. We quantify this statement in our main result.

Proposition 2.6. There are constants $\gamma(n) \in \mathbb{R}$ such that for all $A, E \in M_n(\mathbb{R})$, A regular and $E \geq 0$, there holds

$$\frac{1}{\rho(|A^{-1}| \cdot E)} \leq \sigma(A, E) \leq \frac{\gamma(n)}{\rho(|A^{-1}| \cdot E)}. \tag{22}$$

These constants $\gamma(n)$ satisfy

$$n \leq \gamma(n) \leq (3 + 2 \cdot \sqrt{2}) \cdot n. \tag{23}$$

The left inequality in (23) is sharp. Furthermore, for the constants ψ_n being defined in Theorem 2.4 there holds

$$n \leq \gamma(n) \leq \psi_n \cdot n\ . \tag{24}$$

Proof. By [8], Lemma 6.1 and (8) we know that $\sigma(A, E)$ and $\rho_0^S(A)$ depend continuously on the entries of A, E. Hence, we may assume w.l.o.g. $E > 0$ and therefore $|A^{-1}| \cdot E > 0$. Let $x > 0$ be the right Perron vector of $|A^{-1}| \cdot E$. For a regular and nonnegative diagonal matrix $D \in M_n(\mathbb{R})$ there holds (cf. [1]) $\sigma(A, E) = \sigma(AD, ED)$, and $\rho(|(AD)^{-1}| \cdot ED) = \rho(|A^{-1}| \cdot E) =: \rho$. Defining diagonal $D \in M_n(\mathbb{R})$ by $D_{ii} := x_i^{-1}$ shows that we may assume w.l.o.g.

$$\{|A^{-1}| \cdot E\} \cdot (\mathbf{1}) = \rho \cdot (\mathbf{1}).$$

Furthermore, $|A^{-1}| \cdot E > 0$ implies $\rho > 0$ and therefore $\rho^{-1} \cdot |A^{-1}| \cdot E$ is row stochastic. Applying Lemma 2.5 yields

$$\sigma(A, E) \cdot \rho(|A^{-1}| \cdot E) \leq n \cdot \psi_n,$$

and therefore the right inequalities of (23) and (24). The left inequality is contained in [8], Lemma 5.7 together with the fact that it is sharp. The proposition is proved. ■

We mention that in many applications the product $\sigma(A, E) \cdot \rho(|A^{-1}| \cdot E)$ is, for the specific data, not too far from 1. For classes of matrices like M-matrices it is in fact *equal to* 1

([8], (5.5)). From the proof of Lemma 2.5, from Theorem 2.4 and Proposition 2.6 we also conclude the following corollary.

Corollary 2.7. Let $A, E \in M_n(\mathrm{IR})$, A regular and $E \geq 0$, be given. Then for any cycle ω and the constants $\psi_{|\omega|}$ as defined in Theorem 2.4,

$$\frac{1}{\rho(|A^{-1}| \cdot E)} \leq \sigma(A, E) \leq \frac{\psi_{|\omega|}}{\prod(|A^{-1}| \cdot E)_{\omega}^{1/|\omega|}}, \tag{25}$$

where the r.h.s. of (25) becomes at least as small as $n \cdot \psi_n / \rho(|A^{-1}| \cdot E) \leq (3 + 2\sqrt{2}) \cdot n / \rho(|A^{-1}| \cdot E)$ for some ω.

For given data, (25) frequently yields reasonable bounds. Proposition 2.6 shows the asymptotically linear behaviour of

$$\gamma(n) := \sup\{\, \sigma(A, E) \cdot \rho(|A^{-1}| \cdot E) \ \ | \ \ A, E \in M_n(\mathrm{IR}),\ A \text{ regular},\ E \geq 0\}. \tag{26}$$

From (23) we know $n \leq \gamma(n) \leq (3 + 2 \cdot \sqrt{2}) \cdot n$, where the lower bound is sharp. We repeat our conjecture as has been stated in [8].

Conjecture 2.8. For $\gamma(n)$ as defined in (26), there holds $\gamma(n) = n$.

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