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THE SIGN-REAL SPECTRAL RADIUS AND CYCLE PRODUCTS

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Abstract. The extension of the Perron-Frobenius theory to real matrices without sign restriction uses the sign-real spectral radius as the generalization of the Perron root. The theory was used to extend and solve the conjecture in the affirmative ([1], [2]) that an ill-conditioned matrix is nearby a singular matrix also in the componentwise sense. The proof estimates the ratio between the sign-real spectral radius and the maximum geometric mean of a cycle product. In this note we discuss bounds for this ratio including a counterexample to a conjecture about this ratio.

Key words: sign-real spectral radius, Perron-Frobenius theory, componentwise distances.

The sign-real spectral radius for a real matrix $A \in M_n(\mathbf{R})$ is defined by [4]

$$\rho_0^S(A) = \max_{\tilde{S}} \left\{ |\lambda| \; : \; \lambda \text{ } \textit{real} \text{ eigenvalue of } \tilde{S}A \right\} \;,$$

where the maximum is taken over all signature matrices $\tilde{S}$, i.e. diagonal $\tilde{S}$ with $|\tilde{S}_{ii}| = 1$. A set $\omega := \{\omega_1, ..., \omega_k\}$ of $|\omega| := k \geq 1$ mutually distinct integers out of $\{1, ..., n\}$ defines a cycle product

$$\prod A_\omega := A_{\omega_1\omega_2} \cdot ... \cdot A_{\omega_{k-1}\omega_k} \cdot A_{\omega_k\omega_1}$$

in A. The maximum geometric mean of cycle products is defined by

$$\zeta(A) := \max_{\omega} \left| \prod A_\omega \right|^{1/|\omega|} .$$

It is well known that [3, Theorem 5.7.21]

$$\zeta(A) = \inf\{||D^{-1}AD||_\infty \; : \; D \in M_n(\mathbf{R}) \quad \text{positive diagonal matrix}\} \;,$$

where $||A||_\infty := \max |A_{ij}|$.

The following theorem [4, Theorem 5.6] states a two-sided estimation between $\rho_0^S(A)$ and $\zeta(A)$. It is the key to prove that an ill-conditioned matrix cannot be far from a singular matrix in a componentwise sense.

THEOREM 1.1. *Let A be a real $n \times n$ matrix. Then*

$$(3 + 2\sqrt{2})^{-1} \cdot \zeta(A) \leq \rho_0^S(A) \leq n \cdot \zeta(A) \quad . \tag{1.1}$$

The right inequality is sharp for A being the matrix of all ones.

The question remains what are best constants for the left inequality in (1.1), i.e. what is the value of

$$\inf_A \rho_0^S(A)/\zeta(A) \quad ? \tag{1.2}$$

Improving the estimation implies an improvement in the estimation of the ratio of the componentwise distance to the nearest singular matrix and the componentwise

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condition number. In [4, Conjecture 6.1] it has been conjectured that for matrices of the form

$$A = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & * & \\ & & & \ddots & \ddots & \\ & * & & & & 1 \\ 1 & & & & & 0 \end{pmatrix}$$

(* denoting arbitrary real numbers) there exists a nontrivial vector x with $|Ax| \geq |x|$, where absolute value and comparison are to be understand componentwise. The appealing fact about this conjecture is, despite several conclusions which follow, the ease of formulation. The conjecture is shown in [4] to be equivalent to $\rho_0^S(A) \geq \zeta(A)$ for arbitrary matrices A with zero diagonal. Unfortunately, there are counterexamples, at least for $n \geq 6$. Consider

$$B = \text{circulant}(-0.3, 1, -0.8), \quad \text{and } A = \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix},$$

where I denotes the 3×3 identity matrix. It is $\zeta(B) = \zeta(A) = 1$, and a calculation shows $\rho_0^S(B) < 0.95$. For signature matrices S_1, S_2, the eigenvalues of $\text{diag}(S_1, S_2) \cdot A$ are the squares of the eigenvalues of $S_1 B S_2$, and by $\rho_0^S(B) = \rho_0^S(S_1 B S_2)$ [4, Lemma 2.1] it follows $\rho_0^S(A) = [\rho_0^S(B)]^{1/2} < 0.98 < \zeta(A)$.

For an upper bound of (1.2) we use [4, Corollary 2.14]

$$(1.3) \quad \rho_0^S(A) = \min\{b \geq 0 \, : \, \det(bI - SA) \geq 0 \text{ for all signature matrices } S\} \quad .$$

Theorem 1.2. *For $n \geq 2$ define the $n \times n$ Toeplitz matrix $A = A(\alpha)$ by*

$$(1.4) \qquad A = \begin{pmatrix} 1-\alpha & & & \\ & & 2-\alpha & \\ & \ddots & & \\ -\alpha & & & \\ & & & 1-\alpha \end{pmatrix} \quad \text{for} \quad 0 \leq \alpha \leq 2 \, .$$

Then $\rho_0^S(A) = 1$.

Proof. We proceed by induction. For S being a signature matrix and $n = 2$, the only real eigenvalues of SA are -1 and 1 in the specified range of α. Assume the assertion is true for matrices of dimension less than n. According to (1.3) it suffices to prove

$$\begin{array}{ll} \det(I - SA) \geq 0 & \text{for } \textit{all} \text{ signature matrices } S, \text{ and} \\ \det(I - \tilde{S}A) = 0 & \text{for } \textit{some} \text{ signature matrix } \tilde{S}. \end{array}$$

Let S be given. For $S_{11} = 1, S_{22} = -1$, the sum of the first two rows of $I - SA$ is zero, for $S_{11} = -1, S_{22} = 1$, the sum of the first two columns is zero, respectively. Denote the principal submatrix of $I - SA$ obtained by deleting the first row and column by $(I - SA)(1)$. For $S_{11} = 1, S_{22} = 1$, subtract the second column of $I - SA$ from the first column, for $S_{11} = -1, S_{22} = -1$, subtract the second row of $I - SA$ from the first row. In either case $\det(I - SA) = 2 \cdot \det((I - SA)(1))$, and the induction finishes the proof. ∎

THEOREM 1.3. *Let $A := A(\alpha)$ be the matrix defined in (1.4) for $\alpha := 2/n$. Then*

$$\rho_0^S(A)/\zeta(A) = \frac{1}{2} \cdot \frac{n}{n-1} \cdot (n-1)^{1/n} \ .$$

Therefore

$$\inf_A \rho_0^S(A)/\zeta(A) \le 1/2 \ ,$$

where the infimum is taken over all real matrices of arbitrary size.

Proof. The largest cycle product is achieved for the full cycle $\omega = (1, ..., n)$. A computation yields the result. □

By Theorem 1.1, the general bound

$$c \cdot \zeta(A) \le \rho_0^S(A)$$

is true for $c = 1/(3 + 2\sqrt{2})$, and Theorem 1.3 shows that such a general bound requires $c \le 1/2$. Is $c = 1/2$? *

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*Note added in proof: In the meantime, Lixing Han and Miki Neumann conjectured the following: For a real matrix whose entries are bounded by one with a simple cycle of 1's, there exists a nonnegative vector x with $|Ax| \ge \frac{1}{2}x$. Han and Neumann proved the conjecture to be true for $n \le 4$. For $n \ge 5$, much numerical evidence suggests that this is the case.