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PERRON-FROBENIUS THEORY FOR COMPLEX MATRICES

SIEGFRIED M. RUMP *

Abstract. The purpose of this paper is to present a unified Perron-Frobenius Theory for nonnegative, for real not necessarily nonnegative and for general complex matrices. The sign-real spectral radius was introduced for general real matrices. This quantity was shown to share certain properties with the Perron root of nonnegative matrices. In this paper we introduce the sign-complex spectral radius. Again, this quantity extends many properties of the Perron root of nonnegative matrices to general complex matrices. Various characterizations will be given, and many open problems remain.

1. Introduction. The key to the generalizations of Perron-Frobenius Theory to general real and to complex matrices is the following nonlinear eigenvalue problem:

$$\max\{|\lambda|: \quad |Ax| = |\lambda x|,\ x \neq 0\}. \tag{1}$$

Throughout the paper we use the notation that *absolute value and comparison of vectors and matrices is always to be understood componentwise*. For example, for $C \in M_n(\mathbb{C})$ and $A \in M_n(\mathrm{I\!R}),\ |C| \le A :\Leftrightarrow |C_{ij}| \le A_{ij}$ for all i, j.

For nonnegative matrices, we can in (1) clearly omit the absolute values and obtain the well known Perron root (ρ denotes the spectral radius):

$$A \in M_n(\mathrm{I\!R}),\ A \ge 0: \quad \begin{aligned} \rho(A) &= \max\{|\lambda|: |Ax| = |\lambda x|,\ \lambda \in \mathbb{C},\ 0 \neq x \in \mathbb{C}^n\} \\ &= \max\{0 \le \lambda \in \mathrm{I\!R}: Ax = \lambda x,\ 0 \le x \in \mathrm{I\!R}^n,\ x \neq 0\}. \end{aligned} \tag{2}$$

For the extension to general real matrices, we purposely restrict attention to real eigenvalues (and eigenvectors), that is we consider the quantity

$$A \in M_n(\mathrm{I\!R}): \quad \max\{|\lambda|: \quad |Ax| = |\lambda x|,\ \lambda \in \mathrm{I\!R},\ 0 \neq x \in \mathrm{I\!R}^n\}. \tag{3}$$

This quantity was introduced and investigated as the sign-real spectral radius $\rho_0^{\mathfrak{S}}(A)$ in [20]. Over there we used another equivalent definition.

For general complex matrices we consider the quantity

$$A \in M_n(\mathbb{C}): \quad \max\{|\lambda|: \quad |Ax| = |\lambda x|,\ \lambda \in \mathbb{C},\ 0 \neq x \in \mathbb{C}^n\}. \tag{4}$$

This was introduced and investigated in our talk in Oberwolfach as the sign-complex spectral radius $\rho^{\mathfrak{T}}(A)$.

In the following we will change the notation of the three quantities (2), (3) and (4) into $\rho^{\mathrm{I\!R}_+}, \rho^{\mathrm{I\!R}}$ and $\rho^{\mathbb{C}}$ to underline the similarities and to emphasize the extension of Perron-Frobenius Theory.

A real (complex) diagonal matrix S with diagonal entries of modulus one is called a real (complex) signature matrix, respectively. Real (complex) signature matrices are the set of diagonal orthogonal (unitary) matrices, which are in the real case the 2^n matrices with diagonal entries ± 1. In our entrywise notation of absolute value, real and complex signature matrices S are characterized by $|S| = I$, I denoting the identity matrix.

For a real or complex vector x, that is $x \in \mathrm{I\!K}^n$ for $\mathrm{I\!K} \in \{\mathrm{I\!R}, \mathbb{C}\}$, there is always a signature matrix $S \in M_n(\mathrm{I\!K})$ with $Sx = |x|$. If all entries of x are nonzero, S is unique. Hence, for our nonlinear eigenvalue problem (1) there are signature matrices S_1 and S_2 with $S_1 Ax = |Ax|$ and $S_2 \lambda x = |\lambda x|$, such that

$$|Ax| = |\lambda x| \text{ is equivalent to } S_1 Ax = S_2 \lambda x. \tag{5}$$

* Inst. f. Informatik III, Technical University Hamburg-Harburg, Schwarzenbergstr. 95, 21071 Hamburg, Germany

Note this is true in the real and in the complex case. Therefore the quantity in (3) is for $A \in M_n(\mathrm{IR})$ and $S = S_2^T S_1$ the same as

$$\max\{|\lambda| : \quad SAx = \lambda x,\ \lambda \in \mathrm{IR},\ 0 \neq x \in \mathrm{IR}^n,\ S \in M_n(\mathrm{IR}),\ |S| = I\},$$

and the quantity in (4) is for $A \in M_n(\mathbf{C})$ and $S := S_2^* S_1$ the same as

$$\max\{|\lambda| : \quad SAx = \lambda x,\ \lambda \in \mathbf{C},\ 0 \neq x \in \mathbf{C}^n,\ S \in M_n(\mathbf{C}),\ |S| = I\}.$$

The difference is just the space of the involved quantities λ, x and S. And this unified view also extends to the third quantity, the Perron root (2), because there is exactly one nonnegative real signature matrix, namely the identity matrix, and the Perron vector and the Perron root are known to be nonnegative.

This leads us to the following unified definition of the three quantities (2), (3) and (4).

Definition 1.1. *For* $\mathrm{IK} \in \{\mathrm{IR}_+, \mathrm{IR}, \mathbf{C}\}$ *and* $A \in M_n(\mathrm{IK})$,

$$\rho^{\mathrm{IK}}(A) := \max\{|\lambda| : \quad SAx = \lambda x,\ \lambda \in \mathrm{IK},\ 0 \neq x \in \mathrm{IK}^n,\ S \in M_n(\mathrm{IK}),\ |S| = I\},$$

where $\mathrm{IR}_+ := \{x \in \mathrm{IR} :\ x \geq 0\}$ *denotes the set of nonnegative (real) numbers.*

For $A \in M_n(\mathrm{IR})$, an argument shows that the set on the right hand side is always nonempty (cf., for example, [20, Lemma 2.2]). Note that ρ^{IK} is only defined for $A \in M_n(\mathrm{IK})$. Especially, for nonnegative matrices all three quantities are defined - and are all equal to the Perron root, that is

$$\rho^{\mathrm{IR}_+}(A) = \rho^{\mathrm{IR}}(A) = \rho^{\mathbf{C}}(A) = \rho(A) \quad \text{for nonnegative } A. \tag{6}$$

In previous notation, $\rho^{\mathrm{IR}}(A) = \rho_0^{\mathfrak{S}}(A)$ for real A and $\rho^{\mathbf{C}}(A) = \rho^{\mathfrak{T}}(A)$ for complex A, where $\rho^{\mathrm{IR}_+}(A) = \rho(A)$ for nonnegative A is the Perron root, equal to the (usual) spectral radius. We note that the index zero in $\rho_0^{\mathfrak{S}}$ referred to Rohn's definition of the real spectral radius of a real matrix [18], which is $\rho_0(A) := \max\{|\lambda| :\ \lambda \text{ real eigenvalue of } A\}$, and $\rho_0(A) := 0$ if the spectrum of A is purely complex. It easily follows that

$$\rho^{\mathrm{IR}}(A) = \max\{\rho_0(SA) : \quad |S| = I\},$$

the definition of $\rho_0^{\mathfrak{S}}(A)$ in [20].

Since $\rho^{\mathrm{IR}}[= \rho_0^{\mathfrak{S}}]$ has been called the sign-real spectral radius, we call $\rho^{\mathbf{C}}$ the sign-complex spectral radius. We may use a second signature matrix in Definition 1.1 to restrict x and λ to the nonnegative orthant. For $S_1 Ax = |Ax|$ and $S_2 x = |x|$,

$$|Ax| = |\lambda x| \text{ is equivalent to } S_1 A S_2^* |x| = |\lambda|\,|x|,$$

so that for $\mathrm{IK} \in \{\mathrm{IR}_+, \mathrm{IR}, \mathbf{C}\}$ and $A \in M_n(\mathrm{IK})$,

$$\rho^{\mathrm{IK}}(A) = \max\{0 \leq \lambda \in \mathrm{IR} : \quad S_1 A S_2 x = \lambda x,\ 0 \leq x \in \mathrm{IR}^n,\ S_1, S_2 \in M_n(\mathrm{IK}),\ |S_1| = |S_2| = I\}. \tag{7}$$

The difference in the three definitions is now just the space of the signature matrices S_1 and S_2.

Following, certain properties of the sign-complex spectral radius will be proved. In order to show the similarities between the three quantities $\rho^{\mathrm{IK}}(A)$, $\mathrm{IK} \in \{\mathrm{IR}_+, \mathrm{IR}, \mathbf{C}\}$, namely

(8)
the Perron root $\rho^{\mathrm{IR}_+}(A) = \rho(A)$ for nonnegative matrices,
the sign-real spectral radius $\rho^{\mathrm{IR}}(A)$ for general real matrices, and
the sign-complex spectral radius $\rho^{\mathbf{C}}(A)$ for general complex matrices,

many of the following theorems will be formulated for all three quantities (8). Frequently, the property is identical for all ρ^{IK} and $A \in M_n(\mathrm{IK})$, underlining the unifying aspects.

Most of such properties of the Perron root are well known, and most properties of the sign-real spectral radius have been shown in [20], for some of them we give simpler proofs. We choose to repeat some of those known results to collect and emphasize the similarities.

The outline of the paper is as follows. In Section 2 we list several basic properties and characterizations of the three quantities (8). Following, certain lower and upper bounds depending on minors and cycle products are given. This proves relations to the componentwise distance to the nearest singular matrix, elaborated in Section 4. We show relations to the structured singular value, and in Section 6 we explore ratios between the three quantities (8). In the concluding remarks in Section 7 we mention several open problems.

2. Properties and characterizations.

We start with some basic observations concerning the sign-complex spectral radius. Throughout the paper quantities S, S_1, S_2 etc. are reserved for signature matrices.

LEMMA 2.1. *Let* $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$, $A \in M_n(\mathbb{K})$, *and let signature matrices* $S_1, S_2 \in M_n(\mathbb{K})$, *a permutation matrix* P, *and a nonsingular diagonal matrix* $D \in M_n(\mathbb{K})$ *be given. Then*

$$\begin{aligned}
\rho^{\mathbb{K}}(A) &= \rho^{\mathbb{K}}(S_1 A S_2) = \rho^{\mathbb{K}}(A^*) = \rho^{\mathbb{K}}(P^T A P) = \rho^{\mathbb{K}}(D^{-1} A D), \\
\rho^{\mathbb{K}}(AD) &= \rho^{\mathbb{K}}(DA), \\
\rho^{\mathbb{K}}(\alpha A) &= |\alpha| \rho^{\mathbb{K}}(A) \qquad \text{for } \alpha \in \mathbb{K}.
\end{aligned}$$

For the Kronecker product $\otimes$ *and* $B \in M_n(\mathbb{K})$ *we have* $\rho^{\mathbb{K}}(A)\rho^{\mathbb{K}}(B) \le \rho^{\mathbb{K}}(A \otimes B)$. *If the permutational similarity transformation putting* $|A|$ *into its irreducible normal form [9, Section 8.3] is applied to* A, *and* $A_{(\nu,\nu)}$ *are the diagonal blocks, then*

$$\rho^{\mathbb{K}}(A) = \max_{\nu} \rho^{\mathbb{K}}(A_{(\nu,\nu)}).$$

Especially, for lower or upper triangular A,

$$\rho^{\mathbb{K}}(A) = \max_i |A_{ii}|.$$

Furthermore, $\rho(A) = \rho^{\mathbb{R}_+}(A) = \rho^{\mathbb{R}}(A) = \rho^{\mathbb{C}}(A)$ *for* $0 \le A \in M_n(\mathbb{R})$.

Proof. The key is the maximization over all signature matrices in $M_n(\mathbb{K})$ in Definition 1.1 or, equivalently, in (7). Then observe $S^* = S^{-1}$, so the eigenvalues of $S_1 A S_2$, $S_2 S_1 A$ and ${S_2}^* A^* {S_1}^*$ are the same, and so are the eigenvalues of $SP^T AP$ and $PSP^T A$, where PSP^T is again a signature matrix. Furthermore, signature matrices and diagonal matrices commute. The eigenvalues of $(S_1 A) \otimes (S_2 B) = (S_1 \otimes S_2)(A \otimes B)$ are the products of the eigenvalues of $S_1 A$ and $S_2 B$, and the rest follows easily. ∎

We mention that it was shown in [28] that $F(A) = P^T D^{-1} S A^{(T)} D P$ are the only linear invertible operators preserving the sign-real spectral radius $\rho^{\mathbb{R}}$. For a real matrix A, the three quantities (8) are always related by

$$\rho^{\mathbb{R}}(A) \le \rho^{\mathbb{C}}(A) \le \rho(|A|) \quad \text{and} \quad \rho(A) \le \rho^{\mathbb{C}}(A) \quad \text{for } A \in M_n(\mathbb{R}). \tag{9}$$

Note that $\rho(A) \le \rho^{\mathbb{R}}(A)$ need not be true because $\rho^{\mathbb{R}}(A)$ maximizes only *real* eigenvalues of SA, $|S| = I$. An example is the matrix defined in (29) for $n \ge 3$. The ratio between the quantities and $\rho(|A|)$ is finite; we come to that in Section 6. There is no immediate relation between $\rho^{\mathbb{K}}(AB)$ and $\rho^{\mathbb{K}}(BA)$. Consider

$$A = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ with } AB = (0), \text{ and } BA = 2A,$$

such that $\rho^{\mathbb{K}}(AB) = \rho(|AB|) = 0$ and $\rho^{\mathbb{K}}(BA) = \rho(|BA|) = 4$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Possible relations between $\rho^{\mathbb{C}}(A \circ A)$, $\rho^{\mathbb{C}}(A^2)$ and $\rho^{\mathbb{C}}(A)^2$ will be investigated in Section 6. Moreover, all three quantities (8) depend continuously on the matrix components, a property which is not so obvious for the sign-real spectral radius [20, Corollary 2.5].

For a first unified characterization of the three quantities (8) we prove the subsequent Theorem 2.4. For the proof we use the following result by Doyle, for which he gave a surprisingly simple proof [5, Lemma 1].

LEMMA 2.2. *(Doyle) For a multivariate polynomial $P \in \mathbb{C}[z_1, \ldots, z_n]$ define*

$$\alpha := \min\{\|z\|_\infty : P(z) = 0\}.$$

Then there exists some $u \in \mathbb{C}^n$ with $P(u) = 0$ and $|u_i| = \alpha$ for $1 \le i \le n$.

We first show how every nontrivial vector implies a lower bound for our three quantities (8).

LEMMA 2.3. *For $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$, $A \in M_n(\mathbb{K})$ and $x \in \mathbb{K}^n$ the following is true:*

$$|Ax| \ge |rx| \to \rho^{\mathbb{K}}(A) \ge |r|. \tag{10}$$

Proof. For $\mathbb{K} = \mathbb{R}_+$ this is a well known fact from Perron-Frobenius Theory [2], where, of course, the absolute values may be omitted. For $\mathbb{K} = \mathbb{R}$ it was proved in [20, Theorem 3.1]. Let $\mathbb{K} = \mathbb{C}$. The assumption implies $S_1 Ax \ge S_2 rx$ for some $|S_1| = |S_2| = I$ and therefore existence of $D \in M_n(\mathbb{C}), |D| \le I$ with $DAx = rx$. Regarding $\det(rI - DA)$ as a complex polynomial in the n unknowns $D_{\nu\nu}$, Lemma 2.2 implies existence of diagonal $\widetilde{D} \in M_n(\mathbb{C})$ with $|\widetilde{D}_{\nu\nu}| = \alpha \le 1$ for all ν and $\det(rI - \widetilde{D}A) = 0$. If $\alpha = 0$ then $r = 0$ and (10) is true. Suppose $\alpha \ne 0$. Then $\det(\alpha^{-1}rI - \alpha^{-1}\widetilde{D}A) = 0$ with $|\alpha^{-1}\widetilde{D}| = I$, a signature matrix. Hence Definition 1.1 implies $\rho^{\mathbb{C}}(A) \ge |\alpha^{-1}r| \ge |r|$. ■

Now we can give one of the nice similarities between the three quantities (8) in discussion by extending (10) to a characterization of $\rho^{\mathbb{K}}$.

THEOREM 2.4. *For $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$ and $A \in M_n(\mathbb{K})$ there holds*

$$\rho^{\mathbb{K}}(A) = \max_{0 \ne x \in \mathbb{K}^n} \min_{x_i \ne 0} \left| \frac{(Ax)_i}{x_i} \right|. \tag{11}$$

Proof. Lemma 2.3 implies that the quantity on the right of (11) is a lower bound for $\rho^{\mathbb{K}}(A)$. And by Definition 1.1 there exists a signature matrix $S \in M_n(\mathbb{K})$ with $SAx = \lambda x, 0 \ne x \in \mathbb{K}^n$ and $|\lambda| = \rho^{\mathbb{K}}(A)$, henceforth $|\frac{(Ax)_i}{x_i}| = \rho^{\mathbb{K}}(A)$ for all i with $x_i \ne 0$. This proves the theorem. ■

To our knowledge, the result for $\mathbb{K} = \mathbb{C}$ was first proved, in a different context, by Doyle [5]. Later it was communicated to the author by Bryan Cain [1] with a different proof.

In a certain sense, Theorem 2.4 reveals a philosophy behind our generalization of Perron-Frobenius Theory to general real and complex matrices. In the classical theory, the nonnegative orthant is the generic one. Accordingly, the Perron vector is nonnegative, or in Theorem 2.4 for $\mathbb{K} = \mathbb{R}_+$, the maximization is over nonnegative vectors.

For the sign-real and sign-complex spectral radius we only know that there *exists* an orthant with a desired property. This can be illustrated by rewriting Theorem 2.4 into

$$\rho^{\mathbb{K}}(A) = \max_{\substack{|S|=I \\ S \in M_n(\mathbb{K})}} \max_{0 \le x \in \mathbb{R}^n} \min_{x_i \ne 0} \left| \frac{(ASx)_i}{x_i} \right|.$$

That means, in a certain sense, maximization is performed over all individual orthants. For $\mathbb{K} = \mathbb{R}_+$ the first max, of course, is superfluous: the "orthant" is known in advance. For $\mathbb{K} = \mathbb{R}$ one can calculate $\rho^{\mathbb{R}}(A)$ by maximizing over the finitely many orthants. For $\mathbb{K} = \mathbb{C}$ computation of $\rho^{\mathbb{C}}$ is a continuous maximization problem.

Another example in this spirit is the following. In classical Perron-Frobenius Theory it is well known that increasing an individual component of a nonnegative matrix cannot decrease the spectral radius. Increasing means moving towards $+\infty$, in the direction of the generic nonnegative orthant. For the sign-real spectral radius the same is true in *one* direction, towards $+\infty$ or towards $-\infty$, except that we do not know the direction in advance. And the same is true in the complex case as stated in the following theorem.

THEOREM 2.5. *Let e_i denote the i-th column of the identity matrix, and let $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$ and $A \in M_n(\mathbb{K})$. Then for $i, j \in \{1, \ldots, n\}$ the following is true:*

(*i*) *For $\mathbb{K} = \mathbb{R}_+$*
$$\rho(A + \alpha e_i e_j^T) \geq \rho(A) \qquad \text{for all } \alpha \geq 0.$$
(*ii*) *For $\mathbb{K} = \mathbb{R}$, there exists $s \in \{-1, +1\}$ such that*
$$\rho^{\mathbb{R}}(A + s\alpha e_i e_j^T) \geq \rho^{\mathbb{R}}(A) \quad \text{for all } \alpha \geq 0.$$
(*iii*) *For $\mathbb{K} = \mathbf{C}$, there exists a half space H in $\mathbf{C}$ such that*
$$\rho^{\mathbf{C}}(A + t e_i e_j^T) \geq \rho^{\mathbf{C}}(A) \qquad \text{for all } t \in H.$$

Proof. Let $|Ax| = |rx|$ with $r = \rho^{\mathbb{K}}(A)$ and some $0 \neq x \in \mathbb{K}^n$. Then all three assertions follow by Lemma 2.3 as follows. For $\mathbb{K} = \mathbb{R}_+$ it is $x \geq 0$ and $\alpha \geq 0$ implies

$$(A + \alpha e_i e_j^T)x \geq Ax = |Ax| = |rx|.$$

Similarly, $|(A + s\alpha e_i e_j^T)x| \geq |Ax|$ for some $s \in \{-1, +1\}$ in case $\mathbb{K} = \mathbb{R}$, and for $\mathbb{K} = \mathbf{C}$ we proceed the same way. ∎

Upper bounds for $\rho^{\mathbb{K}}, \mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$ are generally difficult to compute because they imply lower bounds for the componentwise distance to the nearest singular matrix of certain matrices. This will be elaborated in Section 4. Some simple upper bounds on $\rho^{\mathbb{K}}$ are the following. For $1 \leq p \leq \infty$ denote by $\|A\|_p$ the matrix norm induced by the corresponding vector norm $\|\cdot\|_p$.

THEOREM 2.6. *For $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$ and $A \in M_n(\mathbb{K})$,*

$$\begin{array}{lll} \rho^{\mathbb{K}}(A) \leq \|A\|_p & \textit{for} & 1 \leq p \leq \infty, \\ \rho^{\mathbb{K}}(A) = \rho(A) = \|A\|_2 & \textit{if} & \mathbb{K} = \mathbb{R} \textit{ and } A \textit{ is symmetric or,} \\ & \textit{if} & \mathbb{K} = \mathbf{C} \textit{ and } A \textit{ is normal,} \\ \rho^{\mathbb{K}}(A) = 1 & \textit{if} & \mathbb{K} = \mathbb{R} \textit{ and } A \textit{ is orthogonal or,} \\ & \textit{if} & \mathbb{K} = \mathbf{C} \textit{ and } A \textit{ is unitary.} \\ \rho^{\mathbf{C}}(A) = \rho^{\mathbb{R}}(A) & \textit{for} & A \in M_n(\mathbb{R}) \textit{ and } n = 2. \end{array} \tag{12}$$

Proof. By (7), $S_1 A S_2 x = \rho^{\mathbb{K}}(A) \cdot x$ for some $0 \leq x \in \mathbb{R}^n,\ x \neq 0$. Therefore

$$\rho^{\mathbb{K}}(A) \leq \|S_1 A S_2\|_p \leq \|S_1\|_p \|A\|_p \|S_2\|_p = \|A\|_p.$$

For normal or unitary A we have

$$\|A\|_2 = \rho(A) \leq \rho^{\mathbf{C}}(A) \leq \|A\|_2.$$

The same argument can be used for real symmetric matrices because the eigenvalues are real. Real orthogonal matrices have eigenvalues of absolute value 1. By possibly multiplying the first row by -1 we can achieve $\det A = -1$. Then the value of the characteristic polynomial at zero is -1, forcing existence of a positive eigenvalue, which must be 1. For $A \in M_n(\mathbb{R})$ and $n = 2$ either A is triangular, in which case Lemma 2.1 implies $\rho^{\mathbb{R}}(A) = \max |A_{ii}|$ or, there is a signature matrix S and diagonal D such that $B := D^{-1}SAD$ is symmetric. For the (real) eigenvalue λ of B of largest absolute value it follows $|\lambda| = \rho(B) = \|B\|_2 = \rho^{\mathbb{R}}(B) = \rho^{\mathbf{C}}(B)$, and $\rho^{\mathbb{K}}(B) = \rho^{\mathbb{K}}(A)$ by Lemma 2.1. ∎

The first bound in (12) can be arbitrarily weak, as for

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \text{ with } \rho^{\mathbb{K}}(A) = 0 \text{ for all } \mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}, \text{ but } \|A\|_2 = 1.$$

However, in this case also $\rho(|A|) = 0$, and in Theorem 6.3 we show that this is due to an underlying general fact.

Theorem 2.4 has a number of implications, again showing similarities between the three quantities (8) in discussion. We use the notation $A[\mu]$ for the $k \times k$ principal submatrix of A with rows and columns out of the index set $\mu = (\mu_1, \ldots, \mu_k) \subseteq \{1, \ldots, n\}$.

THEOREM 2.7. *The three quantities in (8) are monotone with respect to principal submatrices, i.e., for* $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$ *and* $A \in M_n(\mathbb{K})$,

$$\rho^{\mathbb{K}}(A[\mu]) \le \rho^{\mathbb{K}}(A).$$

Proof. For $|A[\mu]x| = |rx|$ with $\rho^{\mathbb{K}}(A[\mu]) = r$ and $x \in \mathbb{K}^k, k = |\mu|$, the inequality follows by augmenting x by zeros and application of (10). ∎

Another characterization of the three quantities (8) is the following.

THEOREM 2.8. *Let* $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$ *and* $A \in M_n(\mathbb{K})$ *and* $0 < r \in \mathbb{R}$. *Then the following are equivalent.*

(i) $\rho^{\mathbb{K}}(A) < r$.
(ii) $det(rI - DA) \neq 0$ *for every diagonal* $D \in M(\mathbb{K}), |D| \le I$.

Proof. $(i) \Rightarrow (ii)$ Suppose $\det(rI - DA) = 0$ for some $|D| \le I$, $D \in M(\mathbb{K})$ and let $(rI - DA)x = 0$ for $0 \neq x \in \mathbb{K}^n$. Then $|Ax| \ge |DAx| = |rx|$, and Lemma 2.3 implies $\rho^{\mathbb{K}}(A) \ge |r| = r$. $(ii) \Rightarrow (i)$. Suppose $\rho^{\mathbb{K}}(A) = r' \ge r$, then (7) implies $S_1AS_2x = r'x$ for some $S_1, S_2 \in M_n(\mathbb{K})$, $|S_1| = |S_2| = I$ and $0 \le x \in \mathbb{R}^n$. Then $\det(r'I - S_1AS_2) = 0 = \det(r'I - S_2S_1A) = \det(rI - r/r' \cdot S_2S_1A) = \det(rI - DA) = 0$ with $|D| = |r/r' \cdot S_2S_1| \le I$. ∎

In [20, Theorem 2.3] it was shown for real A that

$$\rho^{\mathbb{R}}(A) < r \quad \Leftrightarrow \quad \det(rI - SA) > 0 \text{ for all } |S| = I,$$

which is a finite characterization. For the next generalization recall that $A \in M_n(\mathbb{R})$ is called P-matrix (P_0-matrix) if all minors of A are positive (nonnegative), and $A \in M_n(\mathbf{C})$ is called positive stable if every eigenvalue of A has positive real part.

THEOREM 2.9. *Let* $0 < r \in \mathbb{R}$. *Then*

(i)	*For* $0 \le A \in M_n(\mathbb{R})$:	$\rho(A) < r$	$\Leftrightarrow$	$rI - A$ *is a* P*-matrix*	
			$\Leftrightarrow$	$rI - A$ *is positive stable.*	
(ii)	*For* $A \in M_n(\mathbb{R})$:	$\rho^{\mathbb{R}}(A) < r$	$\Leftrightarrow$	$rI - SA$ *is a* P*-matrix*	*for all real* $\lvert S\rvert = I$.
(iii)	*For* $A \in M_n(\mathbf{C})$:	$\rho^{\mathbf{C}}(A) < r$	$\Leftrightarrow$	$rI - SA$ *is positive stable*	*for all complex* $\lvert S\rvert = I$.

Proof. (i) Follows by [10, Theorem 2.5.3] applied to the Z-matrix $rI - A$.

(ii) was shown in [20, Theorem 2.3].

(iii) Suppose $\rho^{\mathbf{C}}(A) < r$ and $rI - SA$ not positive stable for some $|S| = I$. By $r > 0$ and continuity there exists $0 < \alpha \le 1$ with $rI - \alpha SA$ having a purely imaginary eigenvalue iy. Then $\det((r - iy)I - \alpha SA) = 0 = \det(\alpha^{-1}(r - iy)I - SA)$. By Definition 1.1, $\rho^{\mathbf{C}}(A) = \rho^{\mathbf{C}}(SA) \ge |\alpha^{-1}(r - iy)| \ge |r - iy| \ge r$, a contradiction. If, on the other hand, $rI - SA$ is positive stable for all $|S| = I$, so is $(r + \alpha)I - SA$ for all $\alpha \ge 0$. Therefore, $\det((r + \alpha)I - S_1AS_2) = \det((r + \alpha)I - S_2S_1A) \neq 0$ for all $|S_1| = |S_2| = I$ and all $\alpha \ge 0$, and (7) finishes the proof. ∎

Theorem 2.9 displays a difference in our three quantities (8). For nonnegative A, the structural properties are strong enough for the above relation to class P (and therefore to class M) and to positive stability.

This is no longer true for general real matrices. For

$$A = \begin{pmatrix} 1 & 0.25 & 0 \\ 0 & 1 & 0.25 \\ -0.25 & 0 & 1 \end{pmatrix}$$

all minors of $B := 1.1I - A$ are positive implying $\rho^{\mathbb{R}}(A) < 1.1$, but B is neither inverse positive nor positive stable. In the next section we give another characterization involving P-matrices.

3. Bounds using determinants and cycles. For a lower and upper bound for the three quantities (8) based on determinants we use the following definition.

DEFINITION 3.1. *For real or complex A,*

$$\delta(A) := \max_{\mu} |\det A[\mu]|^{1/|\mu|},$$

where the maximum is taken over all nonempty $\mu \subseteq \{1, \ldots, n\}$.

With this we have the following two-sided bounds.

THEOREM 3.2. *Define $\varphi_n := (2^{1/n} - 1)^{-1}$. Then for $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbf{C}\}$ and $A \in M_n(\mathbb{K})$ we have*

$$\delta(A) \le \rho^{\mathbb{K}}(A) \le \varphi_n \cdot \delta(A).$$

The left and right bounds are sharp in the sense that equality can be achieved for all n. It is $\varphi_n < 1.45n$.

Proof. For $\mathbb{K} = \mathbb{R}$ this was shown in [20, Theorem 4.2], and for nonnegative A, $\rho(A) = \rho^{\mathbb{R}}(A)$. For $\mathbb{K} = \mathbf{C}$, $\mu \subseteq \{1, \ldots, n\}$ and $\lambda_i(A)$ denoting the eigenvalues of A, $\rho^{\mathbf{C}}(A) \ge \rho^{\mathbf{C}}(A[\mu]) \ge \rho(A[\mu]) = \max |\lambda_i(A[\mu])| \ge |\prod \lambda_i(A[\mu])|^{1/|\mu|} = |\det A[\mu]|^{1/|\mu|}$ proves the left inequality. For the right inequality,

$$\det(zI - A) = z^n + \sum_{|\mu|=k\ge 1} (-1)^k \det A[\mu] z^{n-k} =: z^n + R(z) \tag{13}$$

(cf. [12, 2.15]). There are $\binom{n}{k}$ minors $\det A[\mu]$ of size $|\mu| = k$, so that abbreviating $t := \delta(A)$ implies

$$|R(z)| \le \sum_{|\mu|=k\ge 1} |\det A[\mu]||z|^{n-k} \le \sum_{k=1}^{n} \binom{n}{k} t^k |z|^{n-k} = (|z| + t)^n - |z|^n. \tag{14}$$

For $|z| > \varphi_n t$ it follows $t < (2^{1/n} - 1)|z|$ and therefore $(|z| + t)^n < 2|z|^n$. Combining this with (13) and (14) yields

$$|\det(zI - A)| \ge |z|^n - |R(z)| \ge 2|z|^n - (|z| + t)^n > 2|z|^n - 2|z|^n = 0.$$

This implies $\det(zI - A) \ne 0$ for all $|z| > \varphi_n t$ and therefore $\rho(A) \le \varphi_n t = \varphi_n \delta(A)$. Finally, $\delta(A) = \delta(SA)$ finishes the proof of the inequalities. The left inequalities are equalities for the identity matrix. Finally, in [20, p.28] it was shown that for the circulant

$$A = \operatorname{circ}(1, a, a^2, \ldots, a^{n-1}),\ a := 2^{1/n}, \tag{15}$$

a positive matrix, $|\det A[\mu]| = 1$ for all μ. Therefore, (6) implies $\rho^{\mathbb{K}}(A) = \rho(A) = \sum_{i=0}^{n-1} a^i = (a^n - 1)/(a - 1) = \varphi_n$ showing the right inequality to be sharp for A as in (15) and all n. Finally, $2^{1/n} - 1 = e^{(\ln 2)/n} - 1 > (\ln 2)/n > (1.45n)^{-1}$ finishes the proof. ■

Next we can characterize the case that one of the three quantities (8) is zero. Recall a cycle $(\omega_1, \ldots, \omega_k), k \ge 1$, of a matrix A is a subset of $\{1, \ldots, n\}$. A cycle is called nonzero if the product $A_{\omega_1\omega_2} A_{\omega_2\omega_3} \cdots A_{\omega_k\omega_1}$ is nonzero. Note that every $A_{ii} \ne 0$ defines a nonzero cycle $\{i\}$ of length one. A full cycle is a cycle of length n of mutually different ω_i, i.e. a permutation of $(1, \ldots, n)$. A matrix is acyclic iff it is permutationally similar to a strictly upper triangular matrix. Remarkably, the case $\rho^{\mathbb{K}}(A) = 0$ depends only on this graph theoretical property of A.

THEOREM 3.3. *For* $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$ *and* $A \in M_n(\mathbb{K})$ *the following are equivalent:*

i) $\rho^{\mathbb{K}}(A) = 0$.
ii) A is acyclic.
iii) All minors of A are zero.

Proof. The equivalence of *i)* and *iii)* follows by Theorem 3.2. If A has no cycles, then obviously all minors are zero, so it remains to show *iii)* $\Rightarrow$ *ii)*. Suppose A is not acyclic and let $\mu := (\omega_1, \ldots, \omega_k)$ be a nonzero cycle of minimal length. This is a full cycle of $A[\mu]$. Another nonzero full cycle of $A[\mu]$ implies by [7, Lemma 2.1] a common nonzero subcycle, contradicting the minimality of the length of μ. Hence $A[\mu]$ has only one nonzero cycle at all, and this implies $\det A[\mu] \neq 0$. ∎

The NP-hardness to compute $\rho^{\mathbb{R}}$ [20, Corollary 2.9] is reflected in the exponential number of minors in the definition of δ. Another result in this spirit relates $\rho^{\mathbb{R}}$ to P-matrices.

THEOREM 3.4. *For* $A \in M(\mathbb{R})$ *and* $0 < r \in \mathbb{R}$ *not an eigenvalue of A the following is true:*

$$\rho^{\mathbb{R}}(A) < r \quad \Leftrightarrow \quad (rI - A)^{-1}(rI + A) \text{ is a } P\text{-matrix.}$$

This was proved in [20, Theorem 2.13]. Note that in contrast to Theorem 2.9 (*ii*) there is no signature matrix involved in the characterization of $\rho^{\mathbb{R}}$ in Theorem 3.4. It also gives another proof of NP-hardness to compute $\rho^{\mathbb{R}}$ by using an inverse Cayley transform and because checking P-property is NP-hard [3]. We will use Theorem 3.4 to identify the sign-real spectral radius for certain matrices in order to establish bounds for the ratio $\rho^{\mathbb{C}}/\rho^{\mathbb{R}}$ in Section 6. Concerning the sign-complex spectral radius, it is well known that

$$\rho(A) < r \quad \Leftrightarrow \quad (rI - A)^{-1}(rI + A) \text{ is positive stable}$$

because the Cayley transform maps eigenvalues from the (open) unit disc to the (open) right half plane. What is an equivalent condition for $\rho^{\mathbb{C}}(A) < r$ related to the Cayley transform $(rI - A)^{-1}(rI + A)$? We have reasons to conjecture the following.

CONJECTURE 3.5. *For* $r > 0$,

$$\rho^{\mathbb{C}}(A) < r \quad \Leftrightarrow \quad (rI - A)^{-1}(rI + A) \text{ is (positive) } D\text{-stable.}$$

Recall a matrix is called D-stable if DA is positive stable for all positive diagonal D [10, 2.5.7 f.]. If true, this would be a characterization of D-stability, apparently still an open problem. We mention that for nonsingular real diagonal D,

$$\|D^{-1}AD\|_2 < r \quad \Leftrightarrow \quad D^2C^* + CD^2 \quad \text{positive definite,} \tag{16}$$

where $C := (rI - A)^{-1}(rI + A)$. By Theorem 2.6, the left hand side of (16) implies $\rho^{\mathbb{C}}(A) < r$, where the right hand side implies C to be D-stable. Is there a finite characterization of $\rho^{\mathbb{C}}(A) < r$?

For strictly upper triangular, i.e. acyclic A, Theorem 3.3 implies $\rho^{\mathbb{K}}(A) = 0$, but $\|A\|_2 \neq 0$ for $A \neq 0$. One may ask whether existence of a nonzero cycle already implies that the ratio $\|A\|_2/\rho^{\mathbb{K}}(A)$ becomes finite. By the proof of Theorem 3.3 existence of a nonzero cycle implies at least one minor to be nonzero so that $\rho^{\mathbb{K}}(A)$ is nonzero. Indeed, every nonzero cycle establishes an easy-to-compute and very useful lower bound on $\rho^{\mathbb{R}}(A)$ [21, Theorem 4.4].

This result extends to $\rho^{\mathbb{C}}(A)$. The proof carries almost identically over from the real case [21, Theorem 4.4] to the complex case, so we omit the proof. Again, the result displays a similarity between our three quantities (8).

THEOREM 3.6. *For a matrix A and a cycle $\omega = (\omega_1, \ldots, \omega_k) \subseteq \{1, \ldots, n\}, k \geq 1$, define the geometric mean of the cycle product by*

$$|\prod A_\omega|^{1/|\omega|} := |A_{\omega_1\omega_2} \cdot \ldots \cdot A_{\omega_{k-1}\omega_k} \cdot A_{\omega_k\omega_1}|^{1/k},$$

and the maximum of those by

$$\zeta(A) := \max_\omega |\prod A_\omega|^{1/|\omega|}. \tag{17}$$

Then for $\mathbb{K} \in \{\mathbb{R}_+, \mathbb{R}, \mathbb{C}\}$ and $A \in M_n(\mathbb{K})$,

$$(3 + 2\sqrt{2})^{-1} \cdot \zeta(A) \leq \rho^{\mathbb{K}}(A) \leq n \cdot \zeta(A). \tag{18}$$

For $A = I$, $\zeta(A) = 1 = \rho^{\mathbb{K}}(A)$, and for $A = (\mathbf{1})$, $\rho^{\mathbb{K}}(A) = n = n \cdot \zeta(A)$.

For cycles of length 1 or 2 Theorem 3.6 implies

$$\rho^{\mathbb{K}}(A) \geq \sqrt{|A_{ij}A_{ji}|} \quad \text{for all } 1 \leq i, j \leq n.$$

This includes $\rho^{\mathbb{K}}(A) \geq |A_{ii}|$ for all i, which also follows by Theorem 2.7.

Recently, we used Theorem 3.6 to solve an open problem posed in [14], see [22].

When adapting the proof of Theorem 3.6 from the real case [21, Theorem 4.4] to the complex case $\mathbb{K} = \mathbb{C}$ there is much freedom left. However, we did not manage to utilize this freedom to improve the constant $(3 + 2\sqrt{2})$ in Theorem 3.6 for $\mathbb{K} = \mathbb{C}$. We conjecture that in this case the constant can be replaced by 1. Note that for $\mathbb{K} = \mathbb{R}$ the constant $3 + 2\sqrt{2}$ cannot be replaced by a constant greater than 1/2 [?].

4. Relations to the componentwise distance to singularity. The original motivation to introduce and investigate the sign-real spectral radius was the solution of an open problem posed in [4] concerning the componentwise condition number and distance to singularity of a real matrix, cf. [21]. Much of these results carry over to the complex case and give additional insight.

For a nonnegative weight matrix E and real matrix $A \in M_n(\mathbb{R})$, the real componentwise distance to the nearest singular matrix is defined by

$$d_E^{\mathbb{R}}(A) := \min\{0 \leq \alpha \in \mathbb{R} : \exists\, \widetilde{E} \in M(\mathbb{R}), |\widetilde{E}| \leq \alpha E \text{ and } \det(A + \widetilde{E}) = 0\}. \tag{19}$$

If no such α exists, we define the minimum to be $+\infty$. Correspondingly, for a complex matrix $A \in M_n(\mathbb{C})$ the complex componentwise distance to the nearest singular matrix is defined by

$$d_E^{\mathbb{C}}(A) := \min\{0 \leq \alpha \in \mathbb{R} : \exists\, \widetilde{E} \in M(\mathbb{C}), |\widetilde{E}| \leq \alpha E \text{ and } \det(A + \widetilde{E}) = 0\}. \tag{20}$$

For the special choice $E = I$, i.e. only diagonal componentwise perturbations, there is a simple one-to-one correspondence to the three quantities (8). Part (ii) for $\mathbb{K} = \mathbb{R}$ was first proved in [20, Lemma 2.11].

THEOREM 4.1. *The following is true.*

(i) $d_I^{\mathbb{K}}(A^{-1}) = \rho(A)^{-1}$ *for nonsingular* $0 \leq A \in M(\mathbb{R})$ *and* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.
(ii) $d_I^{\mathbb{K}}(A^{-1}) = \rho^{\mathbb{K}}(A)^{-1}$ *for nonsingular* $A \in M(\mathbb{K})$ *and* $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

Proof. Part (i) follows by (6): $\rho(A) = \rho^{\mathbb{R}}(A) = \rho^{\mathbb{C}}(A)$ for $A \geq 0$.

(ii) For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \geq 0$ we have

$$d_I^{\mathbb{K}}(A^{-1}) > r \quad \Leftrightarrow \quad \forall\, D \in M(\mathbb{K}), |D| \leq rI : \det(A^{-1} + D) \neq 0.$$

Now $\det(A) \neq 0$ implies $d_I^{\mathbb{K}}(A^{-1}) > 0$, and by $r^{-1}(A^{-1}+D) = (r^{-1}I + r^{-1}DA)A^{-1}$ it follows

$$d_I^{\mathbb{K}}(A^{-1}) > r \quad \Leftrightarrow \quad \forall\, \widetilde{D} \in M(\mathbb{K}), |\widetilde{D}| \leq I : \det(r^{-1}I + \widetilde{D}A) \neq 0.$$

Now Theorem 2.8 yields

$$d_I^{\mathbb{K}}(A^{-1}) > r \quad \Leftrightarrow \quad \rho^{\mathbb{K}}(A) < r^{-1}.$$

For $r := d_I^{\mathbb{K}}(A^{-1})$, $\det(A^{-1}+D) = 0$ with $|D| = rI$ implies $\det(r^{-1}I + r^{-1}DA) = 0$, and therefore $\rho^{\mathbb{K}}(A) = r^{-1}$. ■

As a corollary we note that $\rho^{\mathbb{K}}(A)$ depends continuously on the entries of A. This is at least not obvious for $\mathbb{K} = \mathbb{R}$.

Lower bounds for $\rho^{\mathbb{K}}$ are obtained for every nontrivial vector by Lemma 2.3, while Theorem 4.1 implies that computation of upper bounds for $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$ is, in general, difficult. This is because singularity of *some* $A^{-1}+D, |D| \leq r^{-1}I$ implies $d_I^{\mathbb{K}}(A^{-1}) \leq r^{-1}$ and therefore $\rho^{\mathbb{K}}(A) \geq r$, while for an upper bound $r \geq \rho^{\mathbb{K}}(A)$, nonsingularity of *every* $A^{-1}+D, |D| \leq r^{-1}I$ has to be verified.

For general nonnegative weight matrix E and complex nonsingular A, we have $A + \widetilde{E} = A(I + A^{-1}\widetilde{E})$, such that -1 is in the spectrum of $A^{-1}\widetilde{E}$. A simple computation using definition (19) and (20) yields

(21) $\quad d_E^{\mathbb{K}}(A) = [\max\{|\lambda| : \lambda \in \mathbb{K} \text{ eigenvalue of } A^{-1}\widetilde{E}, |\widetilde{E}| \leq E\}]^{-1}$ for $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$ and $A \in M_n(\mathbb{K})$.

Note that for $\mathbb{K} = \mathbb{R}$ the maximum is taken only over *real* eigenvalues. Moreover, $\widetilde{E}$ is freely varying over all $\{\widetilde{E} : |\widetilde{E}| \leq E\} = \{\widetilde{E}S : |\widetilde{E}| \leq E, |S| = I\}$, so that (21) implies

$$d_E^{\mathbb{K}}(A) = \{\max_{|\widetilde{E}| \leq E} \rho^{\mathbb{K}}(A^{-1}\widetilde{E}\}^{-1} \text{ for } \mathbb{K} \in \{\mathbb{R}, \mathbf{C}\} \text{ and } A \in M_n(\mathbb{K}).$$

Still the maximum is taken over all matrices $\widetilde{E}$ with $|\widetilde{E}| \leq E$. This can be improved. For this we need a generalization of the Oettli-Prager Theorem [15] to the complex case.

LEMMA 4.2. *Let* $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$, $A \in M_n(\mathbb{K})$, $0 \leq E \in M_n(\mathbb{R})$, $b \in \mathbb{K}^n$, $0 \leq \delta \in \mathbb{R}^n$, *and define*

$$\sum := \{x \in \mathbb{K}^n : \quad (A + \widetilde{E})x = b + \widetilde{\delta},\ |\widetilde{E}| \leq E,\ |\widetilde{\delta}| \leq \delta\}.$$

Then

$$\sum = \{x \in \mathbb{K}^n : \quad |Ax - b| \leq E|x| + \delta\}.$$

Proof. If $(A + \widetilde{E})x = b + \widetilde{\delta}$, then $|Ax - b| = |-\widetilde{E}x + \widetilde{\delta}| \leq E|x| + \delta$. Conversely, suppose $|Ax - b| \leq E|x| + \delta$. Then there are signature matrices $S_1, S_2 \in M_n(\mathbb{K})$ and real diagonal D with $0 \leq D \leq I$ with $S_1(Ax - b) = DES_2x + D\delta$. With $\widetilde{E} := -S_1^* DES_2$ and $\widetilde{\delta} := S_1^* D\delta$ it follows $(A + \widetilde{E})x = b + \widetilde{\delta}$ and $|\widetilde{E}| \leq E,\ |\widetilde{\delta}| \leq \delta$. ■

This theorem is well known for real matrices [25, Theorem III.2.17] to people working in self-validating methods because it characterizes the solution set of an interval linear system. For $[\mathbf{A}] := \{\widetilde{A} : |A - \widetilde{A}| \leq E\}$ forms an interval matrix and $[\mathbf{b}] := \{\widetilde{b} : |b - \widetilde{b}| \leq \delta\}$ forms an interval vector, it follows

$$\sum = \{x : \quad \widetilde{A}x = \widetilde{b},\ \widetilde{A} \in [\mathbf{A}], \widetilde{b} \in [\mathbf{b}]\}.$$

With Lemma 4.2 we obtain a better characterization of $d_E^{\mathbb{K}}(A)$. For $\mathbb{K} = \mathbb{R}$, this characterization of $d_E^{\mathbb{R}}$ is known [18, Theorem 5.1, (C3)]. For $\mathbb{K} = \mathbf{C}$, the definition (20) and Lemma 4.2 imply

$$\begin{aligned} r := d_E^{\mathbf{C}}(A) &= \min\{0 < \alpha \in \mathbb{R} : \quad (A + \widetilde{E})z = 0,\ 0 \neq z \in \mathbf{C}^n,\ |\widetilde{E}| \leq \alpha E\} \\ &= \min\{0 < \alpha \in \mathbb{R} : \quad |Az| \leq \alpha E|z|,\ 0 \neq z \in \mathbf{C}^n\}. \end{aligned} \tag{22}$$

Then there are $S_1, S_2 \in M_n(\mathbf{C})$, $|S_1| = |S_2| = I$, real diagonal D with $0 \le D \le I$ and real $0 \le x \in \mathbb{R}^n$ with

$$S_1 A S_2 x = r D E x. \tag{23}$$

We show that we may replace D in (23) by complex diagonal $\widetilde{D}$ with $|\widetilde{D}| = \beta I$ for some $0 \le \beta \in \mathbb{R}$. Define the complex polynomial $P(u) := \det(S_1 A S_2 - r\,\mathrm{diag}(u) E) \in \mathbf{C}[u_1, \ldots, u_n]$. By (23), $P(D_{11}, \ldots, D_{nn}) = 0$. For $\beta := \min\{\|u\|_\infty : P(u) = 0\}$, Lemma 2.2 implies existence of some $v \in \mathbf{C}^n$ with $P(v) = 0$ and $|v_i| = \beta$ for all i. Then $\mathrm{diag}(v) = \beta S_3$ for a signature matrix $S_3 \in M_n(\mathbf{C})$, $|S_3| = I$. Furthermore, $P(v) = 0$ implies existence of $0 \ne \widetilde{z} \in \mathbf{C}^n$ with

$$S_1 A S_2 \widetilde{z} = r \beta S_3 E \widetilde{z}.$$

Setting $z := S_2 \widetilde{z}$ we have $|Az| = r\beta E|z|$, and the minimality of r as defined in (22) implies $\beta = 1$. Therefore,

$$\begin{aligned} d_E^{\mathbf{C}}(A) &= \min\{\alpha : \ |Az| = \alpha E|z|,\ 0 \ne z \in \mathbf{C}^n\} \\ &= \min\{\alpha : \ S_1 A S_2 z = \alpha E z,\ 0 \ne z \in \mathbf{C}^n,\ |S_1| = |S_2| = I\} \\ &= \min\{\alpha : \ \det(\alpha^{-1} I - S_2^* A^{-1} S_1^* E) = 0,\ |S_1| = |S_2| = I\} \\ &= \{\max_{|S|=I} \rho^{\mathbf{C}}(A^{-1} S E)\}^{-1} \end{aligned} \tag{24}$$

for complex signature matrices $S, S_1, S_2 \in M_n(\mathbf{C})$. Combining our knowledge on the sign-real spectral radius with [18, Theorem 5.1, (C3)] proves (24) to be true also in the real case.

THEOREM 4.3. *Let $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$, nonsingular $A \in M(\mathbb{K})$ and $0 \le E \in M(\mathbb{R})$ be given. Then*

$$d_E^{\mathbb{K}}(A) = \{\max_{|S|=I} \rho^{\mathbb{K}}(A^{-1} S E)\}^{-1}. \tag{25}$$

By Definition 1.1 it follows that for the characterization of $d_E^{\mathbb{K}}(A)$ only knowledge on the spectrum of a certain set of matrices $A^{-1}\widetilde{E}$, $|\widetilde{E}| = E$, is necessary, namely $A^{-1} S_1 E S_2$. In the real case, this set is finite. Is there a finite characterization on $d_E^{\mathbf{C}}(A)$?

Clearly, Theorem 4.1, (ii) is a consequence of Theorem 4.3 for $E = I$. However, the arguments for $E = I$ may give additional insight into the matter.

Finally, we mention another explicit formula for $d_E^{\mathbb{K}}$ expressed by $\rho^{\mathbb{K}}$.

THEOREM 4.4. *Let $0 \le E \in M(\mathbb{R})$. Then*

(i) $\quad d_E^{\mathbb{K}}(A) = \left[\rho\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix}\right]^{-2}$ *for nonsingular $A \in M(\mathbb{R})$, $A^{-1} \ge 0$, $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$.*

(ii) $\quad d_E^{\mathbb{K}}(A) = \left[\rho^{\mathbb{K}}\begin{pmatrix} 0 & E \\ A^{-1} & 0 \end{pmatrix}\right]^{-2}$ *for nonsingular $A \in M(\mathbb{K})$, $\mathbb{K} \in \{\mathbb{R}, \mathbf{C}\}$.*

Proof. (i) is consequence of (6). (ii) follows by the fact that $\pm\sqrt{\lambda}$ are the eigenvalues of $\begin{pmatrix} 0 & A \\ B & 0 \end{pmatrix}$ for λ an eigenvalue of AB, by Theorem 4.3. ∎

We note that part (i) remains true for $\mathrm{rank}(\mathrm{sign}(A^{-1})) = 1$ when replacing A^{-1} by $|A^{-1}|$ in the formula. This is true, for example, for checkerboard sign distribution of A^{-1}.

For the special case $E = (\mathbf{1})$, i.e. $E_{ij} = 1$ for all i, j, that is for absolute perturbations, we can derive an explicit formula for $d_{(\mathbf{1})}^{\mathbb{K}}(A)$. Let e denote a column of $(\mathbf{1})$, the matrix of all $1's$, so that $(\mathbf{1}) = ee^T$. Then Theorem 4.3 and Lemma 2.1 imply

$$\begin{aligned} d_{(\mathbf{1})}^{\mathbf{C}}(A)^{-1} &= \max_{\substack{u,v \in \mathbf{C}^n \\ |u|=|v|=e}} \rho(A^{-1} u v^*) = \max_{\substack{u,v \in \mathbf{C}^n \\ |u|=|v|=e}} |v^* A^{-1} u| \\ &= \max_{\|u\|_\infty = 1} \|A^{-1} u\|_1 = \|A^{-1}\|_{\infty,1}. \end{aligned}$$

For real $A \in M(\mathrm{I\!R})$ and $\mathrm{I\!K} = \mathrm{I\!R}$, the same is true [19]: $d^{\mathrm{I\!R}}_{(\mathbf{1})}(A)^{-1} = \|A^{-1}\|_{\infty,1}$. In this case the vector u, $\|u\|_\infty = 1$, maximizing $\|A^{-1}u\|_1$ is obviously a real vector with components ± 1 and the eigenvalue of maximum absolute value of $A^{-1}uv^T$, which is $|v^T A^{-1} u|$, is real (note that in Definition 1.1 of $\rho^{\mathrm{I\!R}}$ the maximum is taken over real eigenvalues). Therefore, the real and complex distance to singularity of a real matrix subject to absolute perturbations is the same:

$$d^{\mathbf{C}}_{(\mathbf{1})}(A) = d^{\mathrm{I\!R}}_{(\mathbf{1})}(A) \quad \text{for } A \in M_n(\mathrm{I\!R}). \tag{26}$$

Of course, (26) need not to be true for other weight matrices than $E = (\mathbf{1})$.

Following Poljak and Rohn [17] the computation of $d^{\mathrm{I\!R}}_{(\mathbf{1})}(A)$ is NP-hard. We note that this is true for a very specific subclass of real matrices, namely symmetric, strongly diagonally dominant inverse M-matrices. By Theorem 4.4 and (26),

$$d^{\mathrm{I\!R}}_{(\mathbf{1})}(A) = \rho^{\mathrm{I\!R}} \begin{pmatrix} 0 & (\mathbf{1}) \\ A^{-1} & 0 \end{pmatrix}^{-2} = \rho^{\mathbf{C}} \begin{pmatrix} 0 & (\mathbf{1}) \\ A^{-1} & 0 \end{pmatrix}^{-2}$$

for every real matrix A. This proves the following.

THEOREM 4.5. *The computation of $\rho^{\mathbf{C}}(A)$ is NP-hard.*

Originally, the sign-real spectral radius was introduced [20] to solve a conjecture by Demmel [4]: For $A \in M_n(\mathrm{I\!R})$, there are finite constants γ_n such that

$$\frac{1}{\rho(|A^{-1}|\,|A|)} \le d^{\mathrm{I\!R}}_{|A|}(A) \le \frac{\gamma_n}{\rho(|A^{-1}|\,|A|)}. \tag{27}$$

The quantity in the denominator is the optimal componentwise (Bauer-Skeel) condition number achievable by diagonal scaling [4], [24]. Condition (27) means that the componentwise distance to the nearest singular matrix for relative perturbations is inverse proportional to the (componentwise) condition number. We solved this in the affirmative for general weight matrices $E \ge 0$ instead of $|A|$ (part *(ii)* in the following theorem). The same is true in the complex case, and again the formulations are very similar for the three quantities (8). Note that for normwise perturbations, it is well known that the (normwise) distance to the nearest singular matrix is equal to the reciprocal of the (normwise) condition number [8, Theorem 6.5].

THEOREM 4.6. *For $0 \le E \in M_n(\mathrm{I\!R})$ the following is true (0^{-1} is interpreted as ∞).*

(i) $\dfrac{1}{\rho(|A^{-1}|E)} = d^{\mathrm{I\!R}}_E(A) = d^{\mathbf{C}}_E(A)$ *for nonsingular $A \in M(\mathrm{I\!R}), A^{-1} \ge 0$.*

(ii) $\dfrac{1}{\rho(|A^{-1}|E)} \le d^{\mathrm{I\!K}}_E(A) \le \dfrac{(3+2\sqrt{2})n}{\rho(|A^{-1}|E)}$ *for $\mathrm{I\!K} \in \{\mathrm{I\!R}, \mathbf{C}\}$ and nonsingular $A \in M_n(\mathrm{I\!K})$.*

For every n, there exists a matrix $A \in M_n(\mathrm{I\!K})$ with $d^{\mathrm{I\!K}}_{|A|}(A) = \dfrac{n}{\rho(|A^{-1}|\,|A|)}$.

Proof. Part *(i)* follows by (6) and Theorem 4.3 and the well known fact from Perron-Frobenius Theory that $|A| \le B$ implies $\varrho(A) \le \varrho(B)$, for part *(ii)* and $\mathrm{I\!K} = \mathrm{I\!R}$ see [21, Proposition 5.1]. The proof of part *(ii)* for $\mathrm{I\!K} = \mathbf{C}$ is almost identical to the real case and therefore omitted. For the last part we can use the same example as in the real case [21, (25)], namely the symmetric tridiagonal matrix

$$A = \begin{pmatrix} 1 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & \ddots & \ddots & 0 & 1 \\ & & & 1 & s \end{pmatrix} \in M_n(\mathrm{I\!R}) \subseteq M_n(\mathbf{C}) \text{ with } s := (-1)^{n+1}.$$

A computation yields $|A^{-1}|\,|A| = (\mathbf{1}) \in M_n(\mathrm{I\!R})$ and therefore $\rho(|A^{-1}|\,|A|) = n$. The determinant of A is equal to the sum of two full cycles, both being equal to 1. No componentwise relative perturbation less than 100 %, real or complex, can move the determinant into zero, hence $d^{\mathbf{C}}_{|A|}(A) = d^{\mathrm{I\!R}}_{|A|}(A) = 1$. ∎

The upper bound in (*ii*) relies on the lower bound (18) in Theorem 3.6. If the conjecture following Theorem 3.6 is true, then the constant $3+2\sqrt{2}$ in (*ii*) of the preceeding Theorem 4.6 can be replaced by 1 for $\mathbb{K}=\mathbb{C}$, implying two-sided sharp inequalities in this case.

5. Relations to the structured singular value. In [5] the structured singular value, also known as the μ-number, was introduced to analyze feedback systems with structured uncertainties. For an overview see [16]. The definition of the μ-number relies on a fixed block structure $\mathbf{\Delta} \subseteq M_n(\mathbb{C})$ with

$$\mathbf{\Delta} := \{\mathrm{diag}[\delta_1 I_{r_1}, \ldots, \delta_S I_{r_S}, \Delta_{S+1}, \ldots, \Delta_{S+F}] : \delta_i \in \mathbb{C}, \Delta_{S+j} \in M_{m_j}(\mathbb{C})\}$$

For consistency, $\sum r_i + \sum m_j = n$. For such a block structure, the μ-number is defined by [16]

$$\mu_{\mathbf{\Delta}}(A) := [\min\{\|\Delta\|_2 : \ \Delta \in \mathbf{\Delta}, \ \det(I - A\Delta) = 0\}]^{-1}$$

There are two differences to the reciprocal of the componentwise (complex) distance to singularity $d_E^{\mathbb{C}}$. First, the μ-number refers to blockwise perturbations and second, the distance measure is, with respect to these blocks, normwise. Nevertheless, the μ-number establishes a certain link between normwise and componentwise distance to singularity. For $S=0, F=1$, the μ-number is the reciprocal of the traditional normwise distance to singularity, i.e. $\|A^{-1}\|_2^{-1}$. For $S=n, F=0$, the μ-number is the reciprocal of $d_I^{\mathbb{C}}$ because for 1×1 matrices, the spectral norm and modulus coincide. This is also true for arbitrary S, F and $r_i = m_j = 1$ for all i, j. In this case, $\mu_{\mathbf{\Delta}}(A) = \rho^{\mathbb{C}}(A)$ by Theorem 4.1, and all results on the μ-number are valid for the sign-complex spectral radius. This includes some results in Section 2, especially Theorem 2.4 for $\mathbb{K}=\mathbb{C}$.

With Theorem 4.1 we found arguments why computation of upper bounds for $\rho^{\mathbb{K}}$, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, is generally difficult. For the μ-number, substantial work has been done to investigate the upper bound

$$\rho^{\mathbb{C}}(A) \le \inf_{\substack{D \in M(\mathbb{R}_+) \\ D \text{ diagonal}}} \|D^{-1}AD\|_2. \tag{28}$$

The validity of (28) follows by Theorem 2.6 and Lemma 2.1. The right hand side is numerically convenient to compute because $\|e^{-D}Ae^{D}\|_2$ is convex in the D_{ii} for diagonal D [23]. Convex optimization problems can be solved efficiently. For an excellent treatment see [27]. Over there, sharpness of the bound is also characterized, see also [16].

Estimation (28) is generally referred to as "the upper bound" for the μ-number. It proved to be of good quality in practice, frequently being equal to the left hand side. However, at least asymptotically, the ratio between the upper bound and the μ-number is not finite [26]. Using Theorem 3.6 we obtain an upper bound for the ratio to $\rho^{\mathbb{C}}$ as follows. It is well known [6] that

$$\inf\{\max_{i,j} |D^{-1}AD|_{ij} : D \text{ nonsingular diagonal}\} = \max\{|\prod A_\omega|^{1/|\omega|} : \omega \subseteq \{1,\ldots,n\}\} = \zeta(A).$$

Therefore,

$$\inf_D \|D^{-1}AD\|_2 \le n \cdot \zeta(A)$$

and by Theorem 3.6,

$$\rho^{\mathbb{C}}(A) \le \inf_D \|D^{-1}AD\|_2 \le (3+2\sqrt{2})n \cdot \rho^{\mathbb{C}}(A).$$

But earlier, the better factor $\sqrt{n-1}$ for $n \ge 4$ was found ([13], [11]). It is conjectured that the true ratio is $O(\log n)$.

6. Relations between $\rho^{\mathbb{R}}(A)$, $\rho^{\mathbb{C}}(A)$ and $\rho(|A|)$. We start with a class of matrices which proved useful to construct certain examples - and counterexamples. The sign-real and sign-complex spectral radius can be calculated explicitly for those matrices.

THEOREM 6.1. *Define*

$$A = \begin{pmatrix} 0 & & \mathbf{+1} \\ & \ddots & \\ \mathbf{-1} & & 0 \end{pmatrix} \in M_n(\mathbb{R}) \text{ for } n \geq 2, \tag{29}$$

a skew-symmetric matrix with $A_{ij} = \operatorname{sign}(j-i)$, that is all components equal to $+1$ above and equal to -1 below the zero diagonal. Then for all $n \geq 2$ it holds

$$\rho^{\mathbb{R}}(A) = 1 \quad \text{and} \quad \rho^{\mathbb{C}}(A) = \frac{\sin \pi/n}{1 - \cos \pi/n}.$$

Remark. For the real case this was shown in [20, Lemma 5.6]. Here we give a simpler proof. Note that $\rho^{\mathbb{R}}(A) \geq 1$ is easy to see, but $\rho^{\mathbb{R}}(A) = 1$ means that no matrix SA, $|S| = I$, has a real eigenvalue greater than one in absolute value.

Proof. A direct computation shows

$$P = (I-A)^{-1}(I+A) = \begin{pmatrix} 0 & & & & 1 \\ -1 & 0 & & \mathbf{0} & \\ & -1 & \ddots & & \\ & & \ddots & & \\ & \mathbf{0} & & & \\ & & & -1 & 0 \end{pmatrix} \quad \text{for all } n \geq 2, \tag{30}$$

so that $|P|$ is a permutation matrix. It is $\det P = 1$ for all n, so that this skew-circulant is a P_0-matrix for all n. Now Theorem 3.4 together with a continuity argument shows $\rho^{\mathbb{R}}(A) = 1$.

Next we calculate $\rho^{\mathbb{C}}(A)$. For A being normal, Theorem 2.6 implies $\rho^{\mathbb{C}}(A) = \rho(A)$. The characteristic polynomial of P in (30) is $\chi_P(x) = x^n + (-1)^n$, so that the eigenvalues of P are

$$\begin{array}{lll} \exp(2k\pi i/n), & k = 1 \dots n & \text{for } n \text{ odd}, \\ \exp((2k+1)\pi i/n), & k = 1 \dots n & \text{for } n \text{ even}. \end{array}$$

This yields the eigenvalues of $A = (P+I)^{-1}(P-I)$. A little computation for n odd and n even and Theorem 2.6 shows

$$\rho^{\mathbb{C}}(A) = \|A\|_2 = \rho(A) = \frac{\sin \pi/n}{1 - \cos \pi/n} \quad \text{for all } n \geq 2.$$

■

We first consider relations between the Hadamard product $\rho^{\mathbb{K}}(A \circ A)$, where $(A \circ B)_{ij} := A_{ij}B_{ij}$, and $\rho^{\mathbb{K}}(A)^2$ and $\rho^{\mathbb{K}}(A^2)$. For $\mathbb{K} = \mathbb{R}$ those three quantities may be in any order: For A as in (29) and $n = 3$ we have

$$\rho^{\mathbb{R}}(A)^2 = 1 < \rho^{\mathbb{R}}(A \circ A) = 2 < \rho^{\mathbb{R}}(A^2) = 3,$$

and for

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{pmatrix} \tag{31}$$

we have

$$\rho^{\mathbb{R}}(A^2) = 1 < \rho^{\mathbb{R}}(A \circ A) = 2 < \rho^{\mathbb{R}}(A)^2 = 4.$$

In the complex case and again for the matrix defined in (31) the values do not change compared to the real case, that is

$$\rho^{\mathbb{C}}(A^2) = 1 < \rho^{\mathbb{C}}(A \circ A) = 2 < \rho^{\mathbb{C}}(A)^2 = 4,$$

and three inequalities remain. For A as in (29) and $n = 3$ we have $\rho^{\mathbb{C}}(A \circ A) = 2 < \rho^{\mathbb{C}}(A^2) = 3$. The inequality $\rho^{\mathbb{C}}(A)^2 < \rho^{\mathbb{C}}(A^2)$ is only possible if the bound (28) is not sharp. This is because for A scaled such that $\rho^{\mathbb{C}}(A) = \|A\|_2$ it is

$$\rho^{\mathbb{C}}(A^2) \le \|A^2\|_2 \le \|A\|_2 \cdot \|A\|_2 = \rho^{\mathbb{C}}(A)^2, \tag{32}$$

and in case the infimum in (28) is not a minimum, a continuity argument confirms (32). This implies

$$\rho^{\mathbb{C}}(A)^2 \ge \rho^{\mathbb{C}}(A^2) \text{ for } n \le 3,$$

but for the 4×4 matrix defined in [16, Section 9.2] it is $\rho^{\mathbb{C}}(A)^2 < \rho^{\mathbb{C}}(A^2)$. This example can be generalized to $n > 4$.

The inequality $\rho^{\mathbb{C}}(A)^2 < \rho^{\mathbb{C}}(A \circ A)$ is also not possible if the upper bound (28) is sharp. This is because $\rho^{\mathbb{C}}(A) = \|B\|_2$ with $B = D^{-1}AD$ for diagonal D implies

$$\begin{aligned} \rho^{\mathbb{C}}(A \circ A) &= \rho(S(A \circ A)) = \rho(D^{-2}S(A \circ A)D^2) = \rho(S((D^{-1}AD) \circ (D^{-1}AD))) \\ &\le \|S(B \circ B)\|_2 = \|B \circ B\|_2 \le \|B\|_2^2 = \rho^{\mathbb{C}}(A)^2, \end{aligned}$$

where the last inequality follows by [10, Theorem 5.5.1]. Hence,

$$\rho^{\mathbb{C}}(A \circ A) \le \rho^{\mathbb{C}}(A)^2 \quad \text{if (28) is sharp, especially for } n \le 3. \tag{33}$$

In the real case $A \in M(\mathbb{R})$, we have $A \circ A \ge 0$ and

$$\rho^{\mathbb{C}}(A \circ A) = \rho(A \circ A) \le \rho(A)^2 \le \rho^{\mathbb{C}}(A)^2.$$

Is (33) true for general complex A?

The results can be summarized in the following table.

$<$	$\rho^{\mathbb{C}}(A \circ A)$	$\rho^{\mathbb{C}}(A)^2$	$\rho^{\mathbb{C}}(A^2)$
$\rho^{\mathbb{C}}(A \circ A)$		(31)	(29)
$\rho^{\mathbb{C}}(A)^2$	?		[16, Sec. 9.2]
$\rho^{\mathbb{C}}(A^2)$	(31)	(31)	

The references are examples of matrices such that the quantity in the left column is strictly less than the quantity in the top row. For the "?" such an example may only exist for $n \ge 4$ and if the upper bound (28) is not sharp.

Finally we give bounds for the ratios between $\rho^{\mathbb{R}}(A)$, $\rho^{\mathbb{C}}(A)$ and $\rho(|A|)$. An upper bound for $\rho^{\mathbb{C}}/\rho^{\mathbb{R}}$ follows by Theorem 3.2:

$$\rho^{\mathbb{R}}(A) \le \rho^{\mathbb{C}}(A) \le \varphi_n \cdot \delta(A) \le \varphi_n \cdot \rho^{\mathbb{R}}(A). \tag{34}$$

For the matrices given in Theorem 6.1 we have

$$\rho^{\mathbb{C}}(A) = \frac{\sin \pi/n}{1 - \cos \pi/n} \rho^{\mathbb{R}}(A).$$

The power series expansion

$$\frac{\sin x}{1-\cos x} = \frac{2}{x} - \frac{1}{6}x + 0(x^3)$$

yields $\rho^{\mathbb{C}}(A)/n = 2/\pi + 0(n^{-2})$. Together with (34) this proves the following.

THEOREM 6.2. *For $A \in M_n(\mathbb{R})$,*

(35) $$\rho^{\mathbb{R}}(A) \le \rho^{\mathbb{C}}(A) \le (2^{1/n}-1)^{-1} \cdot \rho^{\mathbb{R}}(A) < 1.45n \cdot \rho^{\mathbb{R}}(A).$$

For all n it is

$$\frac{2}{\pi} \cdot n \le \sup_{A \in M_n(\mathbb{R})} \frac{\rho^{\mathbb{C}}(A)}{\rho^{\mathbb{R}}(A)} \le (2^{1/n}-1)^{-1} \le 1.45n.$$

Finally, for $\zeta(A)$ as defined in (17) we have $\zeta(A) = \zeta(|A|)$, and by Theorem 3.6

$$[(3+2\sqrt{2})n]^{-1}\rho(|A|) \le (3+2\sqrt{2})^{-1}\zeta(A) \le \rho^{\mathbb{R}}(A) \le \rho^{\mathbb{C}}(A) \le \rho(|A|).$$

For a Hadamard matrix H with $H^TH = nI$, Theorem 3.2 and Theorem 2.6 imply

$$|\det H|^{1/n} = n^{1/2} \le \rho^{\mathbb{R}}(H) \le \rho^{\mathbb{C}}(H) \le \|H\|_2 \le n^{1/2}.$$

Hence,

$$\rho(|H|)/\rho^{\mathbb{R}}(H) = \rho(|H|)/\rho^{\mathbb{C}}(H) = n^{1/2}.$$

THEOREM 6.3. *For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $A \in M_n(\mathbb{K})$,*

$$[(3+2\sqrt{2})n]^{-1} \cdot \rho(|A|) \le \rho^{\mathbb{K}}(A) \le \rho(|A|).$$

At least for values of n where an $n \times n$ Hadamard matrix exists it is

$$n^{1/2} \le \sup_{A \in M_n(\mathbb{R})} \frac{\rho(|A|)}{\rho^{\mathbb{K}}(A)}.$$

7. Conclusion. The nonlinear eigenequation $|Ax| = |\lambda x|$ was shown to create quantities for general real and complex matrices similar to the Perron root for real nonnegative matrices. We presented a number of results supporting this unification, but many open problems remain. For $A \in M_n(\mathbb{R})$, $C \in M_n(\mathbb{C})$, we conjecture the following.

(36) For $r > 0$: $\quad \rho^{\mathbb{C}}(C) < r \Leftrightarrow (rI - C)^{-1}(rI + C)$ is (positive) D-stable.

(37) $\rho^{\mathbb{C}}(C \circ C) < \rho^{\mathbb{C}}(C)^2$ is not true for $n \ge 4$.

(38) $$n^{-1}\rho(|A|) \le \rho^{\mathbb{R}}(A).$$

(39) $\rho^{\mathbb{R}}(A) \ge \frac{1}{2}|\prod A_\omega|^{1/|\omega|}$ for every cycle $\omega \subseteq \{1, \ldots, n\}$.

(40) $\rho^{\mathbb{C}}(C) \ge |\prod C_\omega|^{1/|\omega|}$ for every cycle $\omega \subseteq \{1, \ldots, n\}$.

(41) $d_E^{\mathbb{R}}(A) \le \dfrac{n}{\rho(|A^{-1}|E)}$ for $0 \le E \in M_n(\mathbb{R})$.

(42) $d_E^{\mathbb{C}}(C) \le \dfrac{n}{\rho(|C^{-1}|E)}$ for $0 \le E \in M_n(\mathbb{R})$.

If true, the inequalities (39), (40), (41) and (42) are best possible. A main open problem for the computation of $\rho^{\mathbb{C}}(A)$ is

Does there exist a finite characterization of $\rho^{\mathbb{C}}(A)$?

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